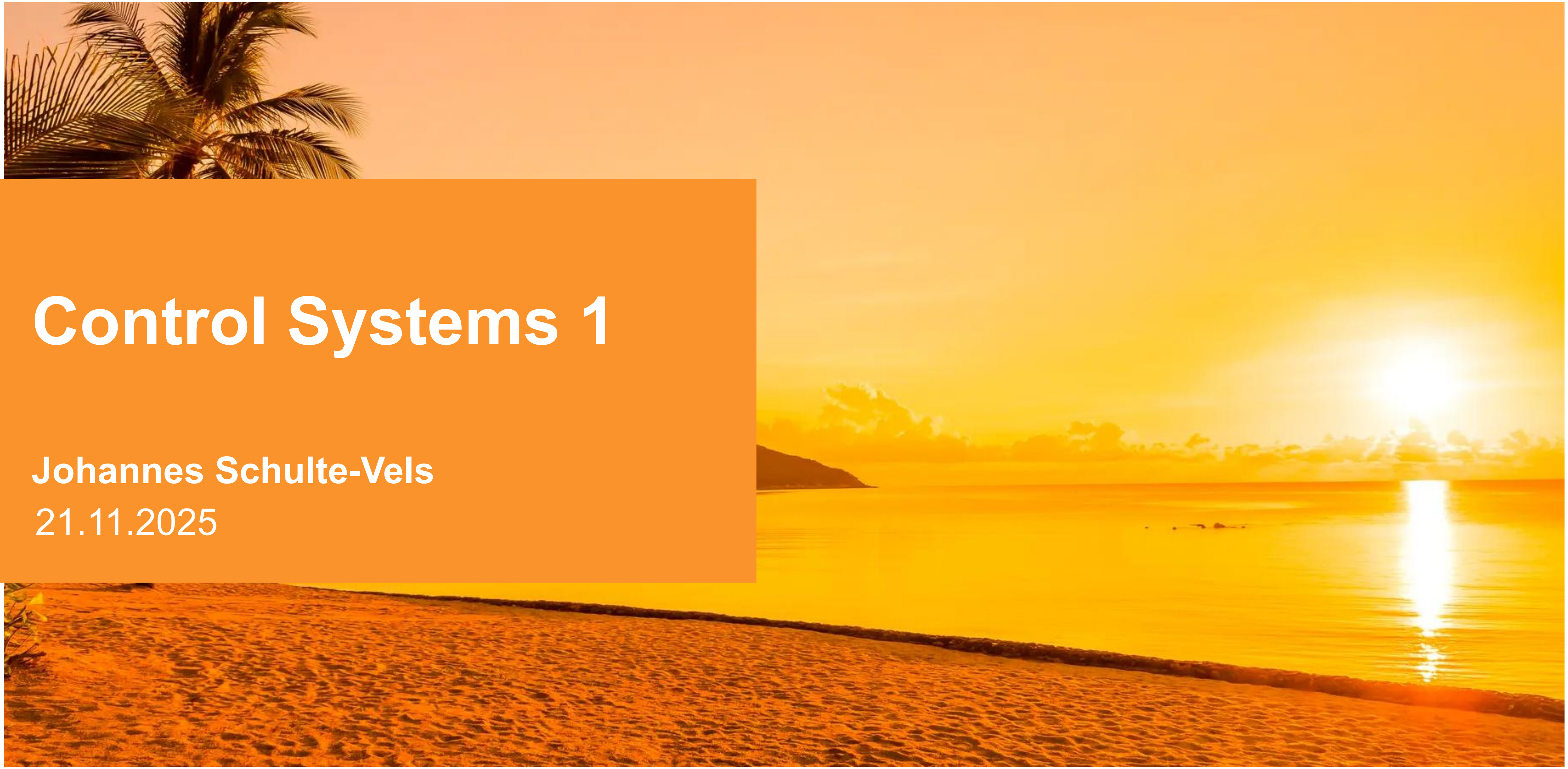
ETH zürich

Institute for Dynamic Systems and Control



Control Systems 1

Johannes Schulte-Vels

21.11.2025

Welcome!

Polybox

PW: jschul

Website

jschultev.github.io/personal_website

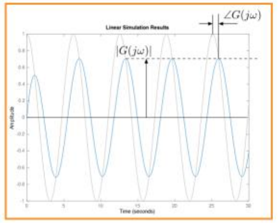
Today

- Repetition Session 9
- Theory Recap
  - Polar Plot
  - Nyquist Plot
  - Nyquist Theorem
  - Nyquist Stability Criterion
  - Frequency Domain Specifications
- Q&A Session / Done

Repetition Session 9

Motivation

$$y_{ss} = |G(j\omega)| \cos(\omega t + \angle G(j\omega))$$

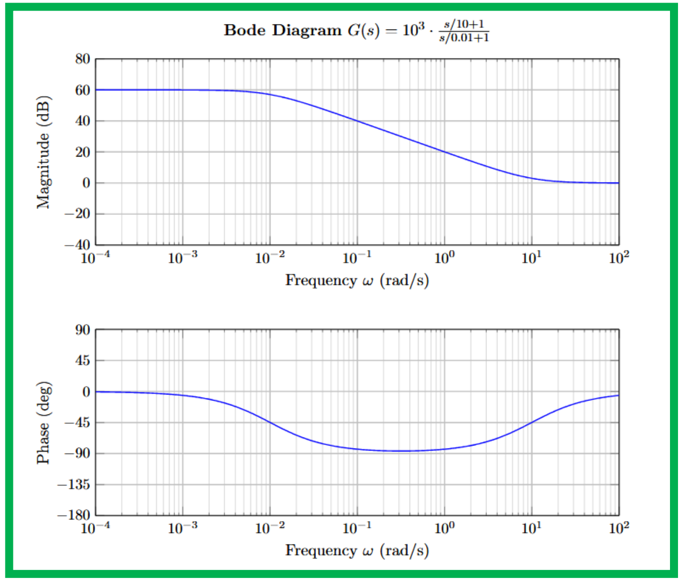


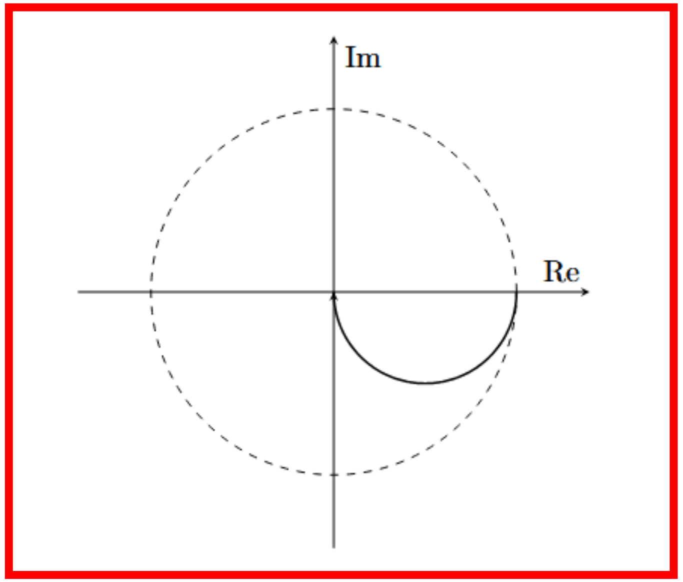
We can see, that magnitude and phase of our TF **do not depend on time t** but only on the **frequency ω**.

Therefore it would be nice to have a plot showing how our frequency response (output) behaves when changing the input frequency. For that we will explore 2 options:

1. **Bode Plot**: The magnitude and the phase of the TF are in 2 seperate plots
2. **Polar / Nyquist Plot**: A parametric curve of the TF with ω implicit

Last week





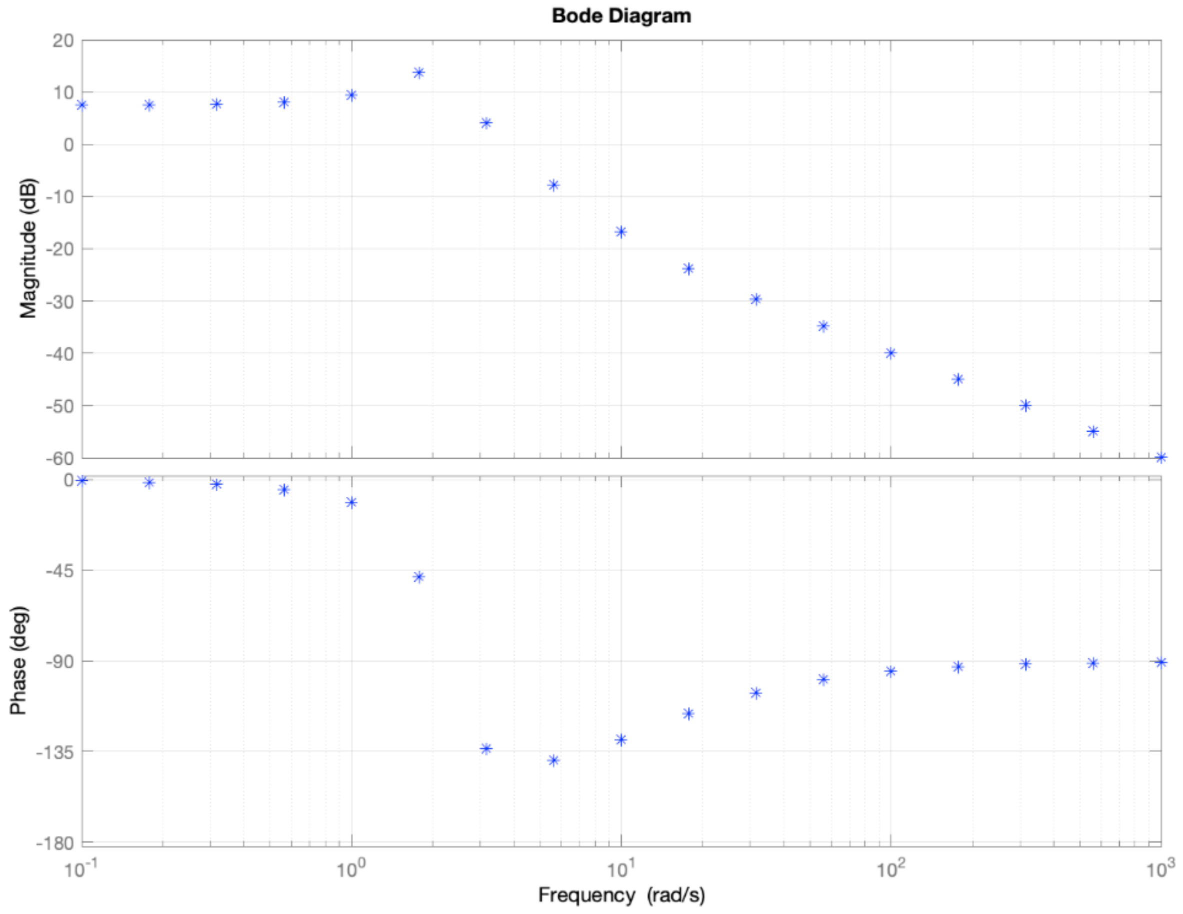
This week

Bode Plot

Let us try out some values for ω for an arbitrary TF and see how the magnitude and phase change when having a sinusoidal input $u(t) = sin(\omega t)$.

We now want to **derive a model with rules** that helps sketch the plot without having to calculate the magnitude and phase for every value of ω …

(similar idea as the root locus for different k)



Bode Plot Rules

1. Write the **TF in Bode form!!**
2. Plot magnitude and phase of **Bode gain**
3. Do not forget to give **magnitude in dB**
4. Superposition all basic compontents. Remember:

   - The **magnitude change** starts **at the position** (frequency) of respective pole or zero
   - The **phase change starts one decade to the left and ends one decade to the right** of the position
   - Integrators and Differentiators have magnitude [dB] = 0 at $\omega = 1$
   - **For complex conjugate** poles remember the **peak and sudden phase change**

1. We are allowed to draw straight lines. Keep in mind however, that this is an approximation.

$$G(s) = \frac{k_{\text{Bode}}}{s^q} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\cdots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\cdots\left(\frac{s}{-p_{n-q}}+1\right)}$$

$$|G(j\omega)|[dB] = 20\log_{10}|G(j\omega)|$$

Term	Magnitude	Phase
Constant K	$20\log_{10}(\lvert K\rvert)$	$\begin{cases} 0^\circ & K>0 \\ \pm 180^\circ & K<0 \end{cases}$
Pole at Origin $\frac{1}{s}$	-20dB/dec	-90° for all ω
Zero at Origin s	$+20$dB/dec	$+90^\circ$ for all ω

Phase \ Magnitude	-20 dB/dec	$+20$ dB/dec
-90°	stable pole	non-minimum phase zero
$+90^\circ$	unstable pole	minimum phase zero

Last Weeks Example

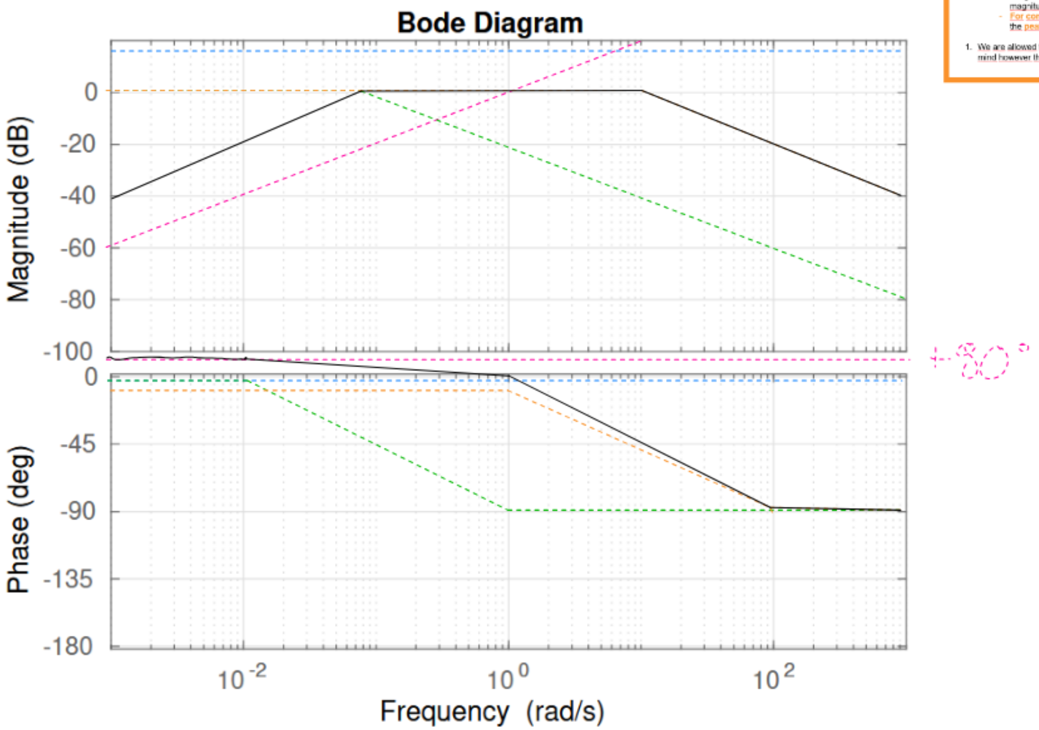
Example

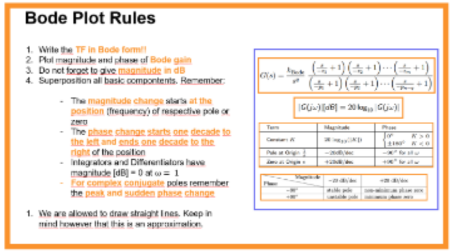
$$G_1(s) = \frac{100s}{(10s+1)(s+10)}$$

$$\frac{100}{10} \cdot \frac{s}{\left(\frac{s}{0.1}+1\right)\left(\frac{s}{10}+1\right)}$$

$= 10$

$20 \cdot \log_{10}(10) = 20$





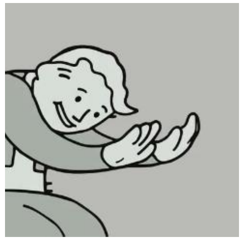
ETH zürich Johannes Schulte-Vels 14.11.2025 40

Inverting a Transfer Function

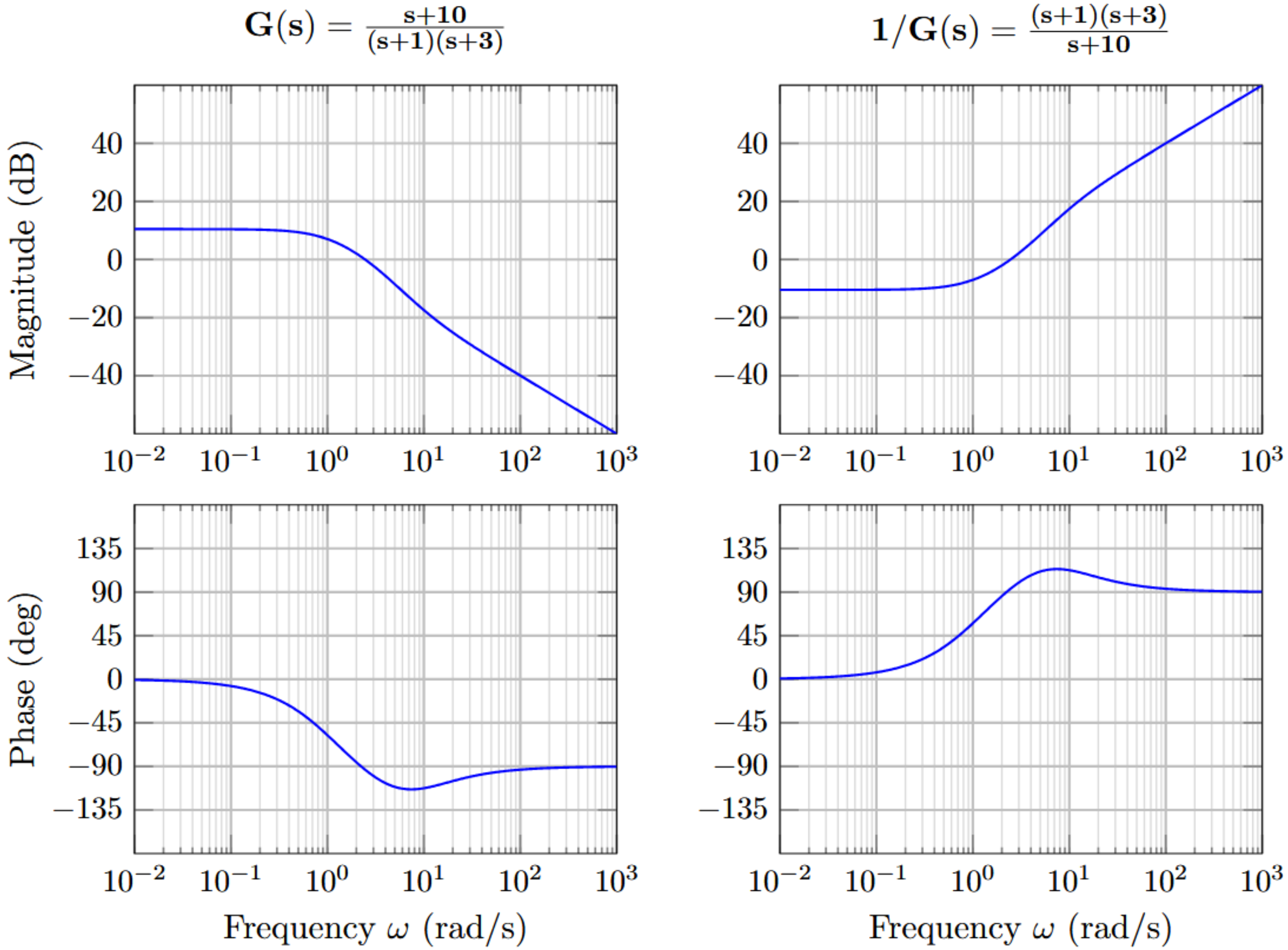
Basically we just mirror the Bode plot on the horizontal axis.

Makes sense, since we can write

$$|1|[\mathrm{dB}] - |G(s)|[\mathrm{dB}] = -|G(s)|[\mathrm{dB}]$$
$$\angle(1) - \angle(G(s)) = -\angle(G(s))$$

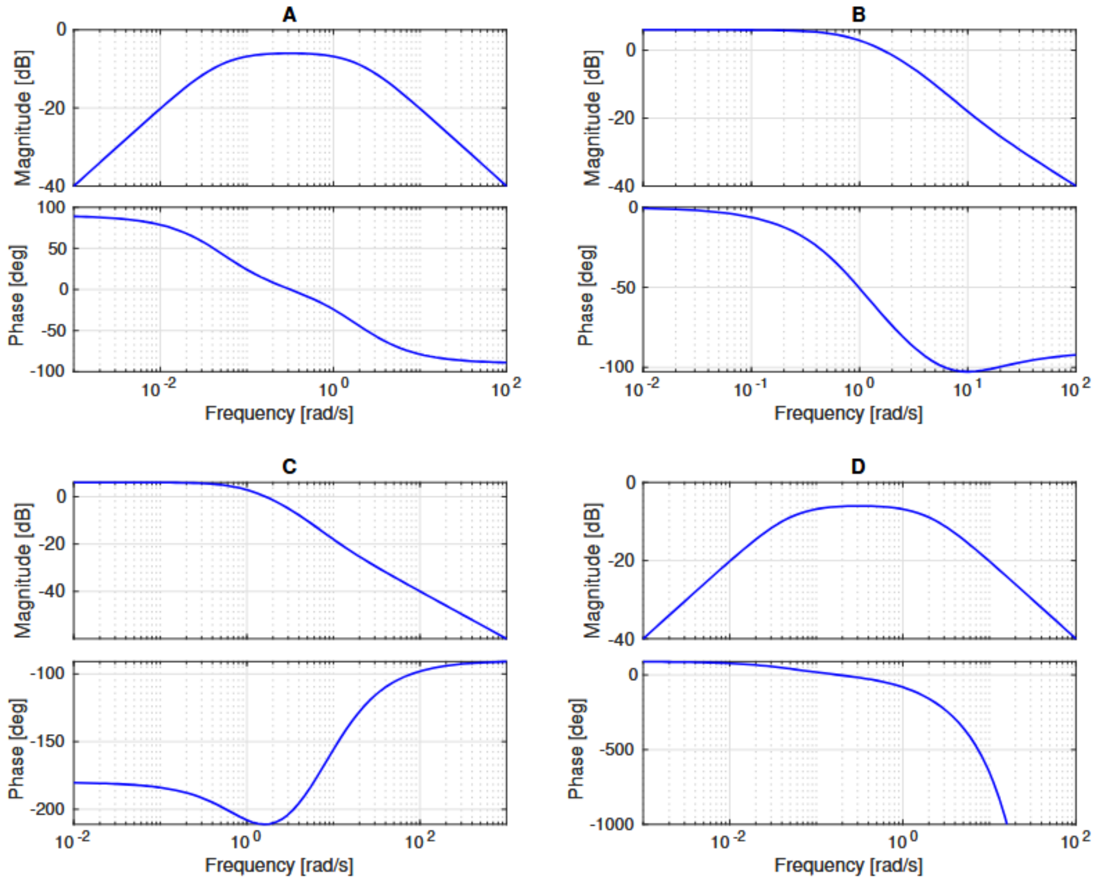


A zero ist just an inverted pole??!!



FS 2024

Phase \ Magnitude	-20 dB/dec	$+20$ dB/dec
$-90°$	stable pole	non-minimum phase zero
$+90°$	unstable pole	minimum phase zero



Match the TF to their bode plots

phase drop a lot

$$G_1(s) = \frac{s}{s^2 + 2s + 0.1} \qquad G_2(s) = e^{-s}\frac{s}{s^2 + 2s + 0.1}$$

$$G_3(s) = \frac{s+10}{(s+1)(s+5)} \qquad G_4(s) = \frac{s+10}{(s+1)(s-5)}$$

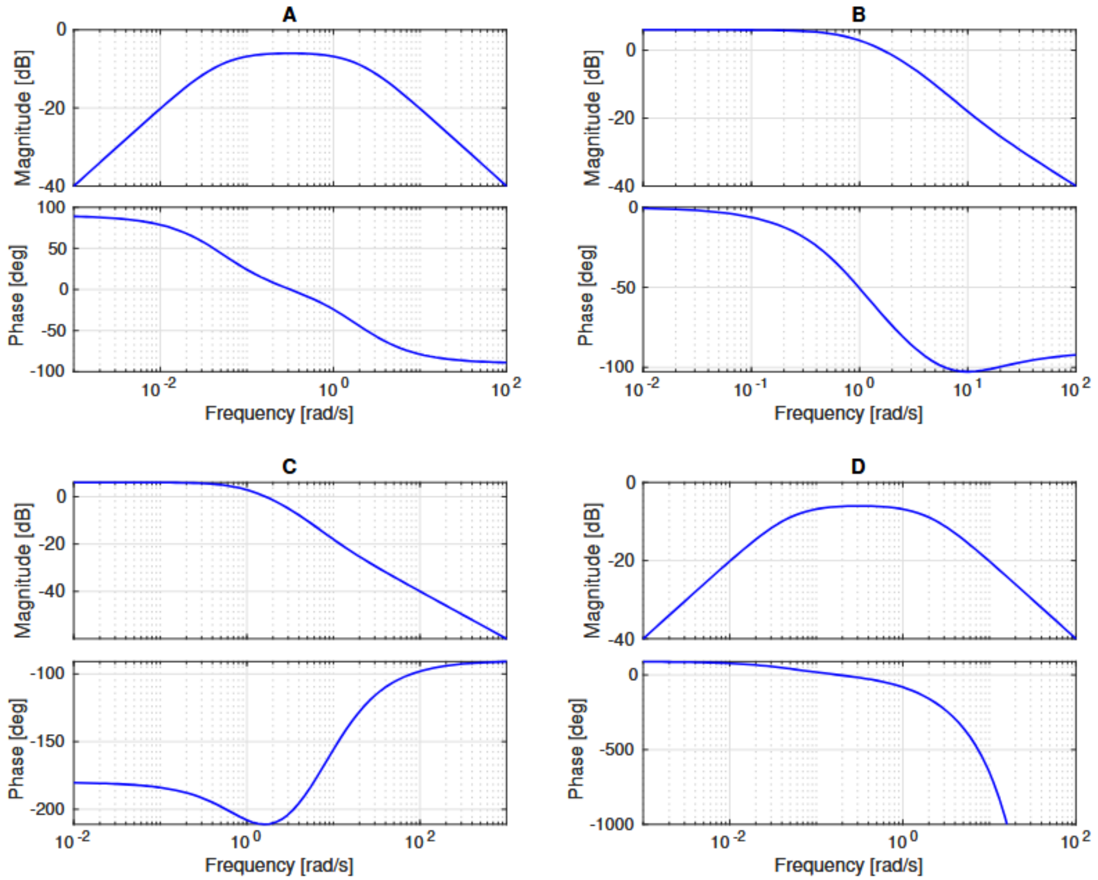
A) $(A, G_1), (B, G_3), (C, G_4), (D, G_2)$

C) $(A, G_2), (B, G_3), (C, G_4), (D, G_1)$

B) $(A, G_1), (B, G_4), (C, G_3), (D, G_2)$

D) $(A, G_3), (B, G_1), (C, G_2), (D, G_4)$

FS 2024



Match the TF to their bode plots

$$G_1(s) = \frac{s}{s^2 + 2s + 0.1} \qquad G_2(s) = e^{-s}\frac{s}{s^2 + 2s + 0.1}$$

$$G_3(s) = \frac{s+10}{(s+1)(s+5)} \cdot \frac{10}{5} \qquad G_4(s) = \frac{s+10}{(s+1)(s-5)} \cdot \frac{10}{-5}$$

neg. gain

A) $(A, G_1), (B, G_3), (C, G_4), (D, G_2)$

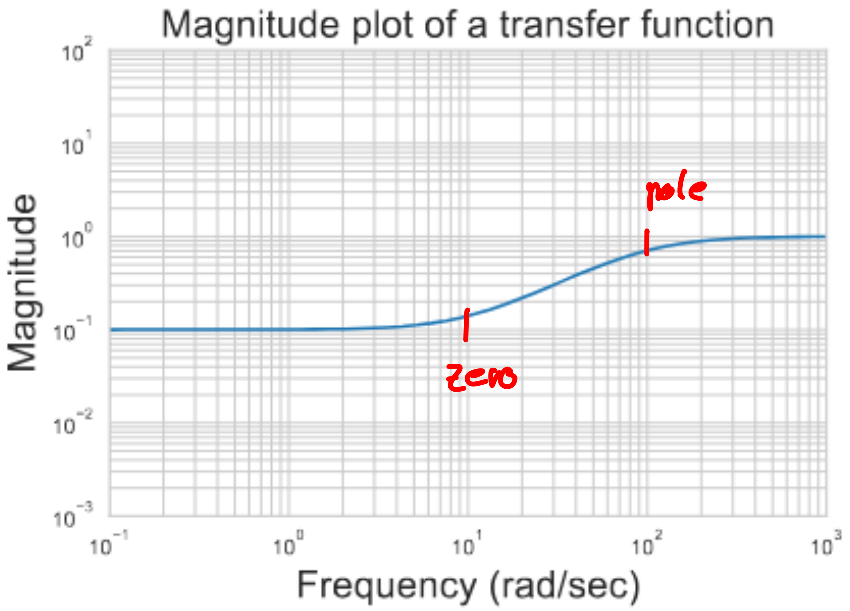
C) ~~$(A, G_2), (B, G_3), (C, G_4), (D, G_1)$~~

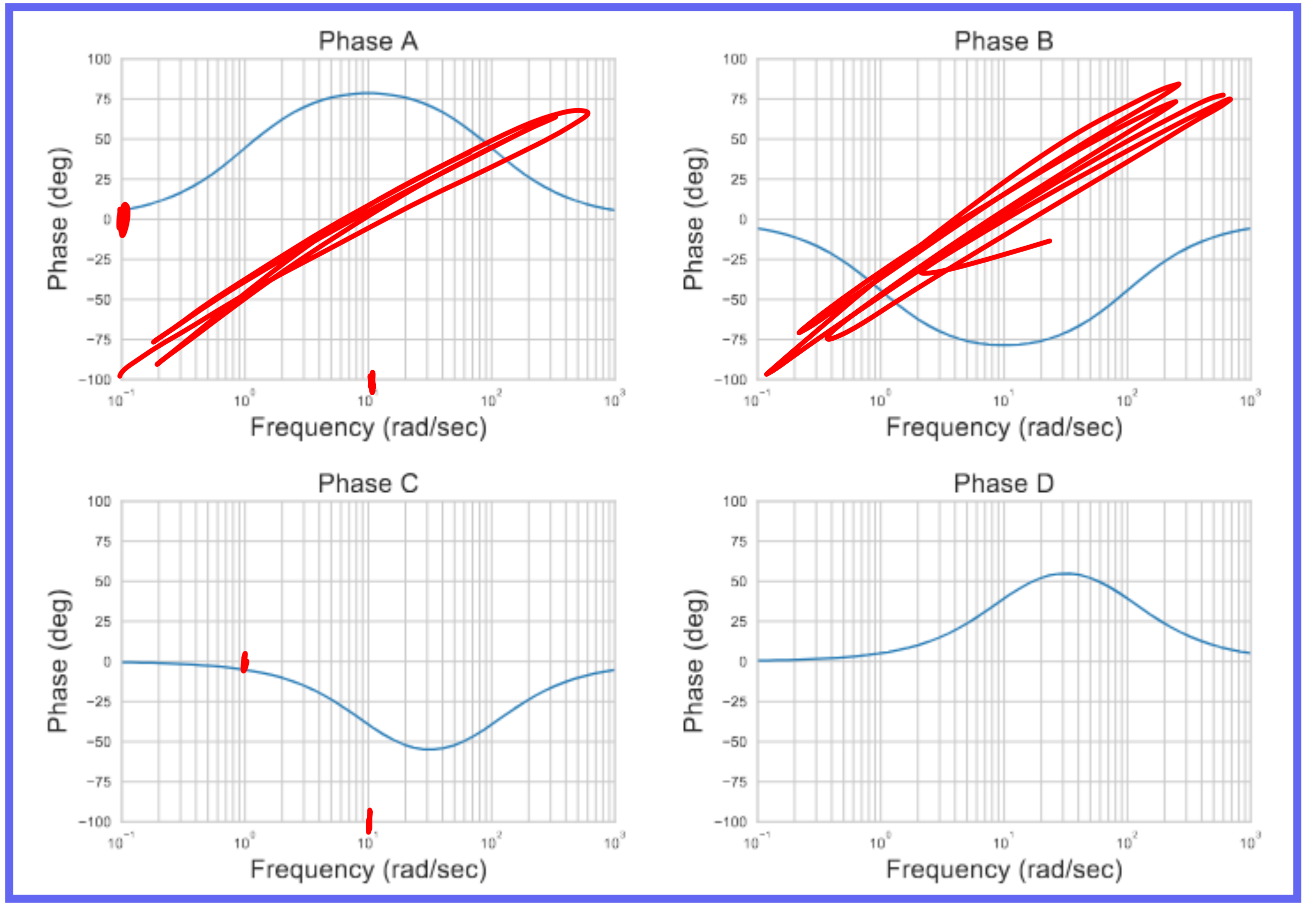
B) $(A, G_1), (B, G_4), (C, G_3), (D, G_2)$

D) ~~$(A, G_3), (B, G_1), (C, G_2), (D, G_4)$~~

FS 2018

Phase \ Magnitude	−20 dB/dec	+20 dB/dec
−90°	stable pole	non-minimum phase zero
+90°	unstable pole	minimum phase zero





Which of the following phase plots corresponds to the given magnitude plot?

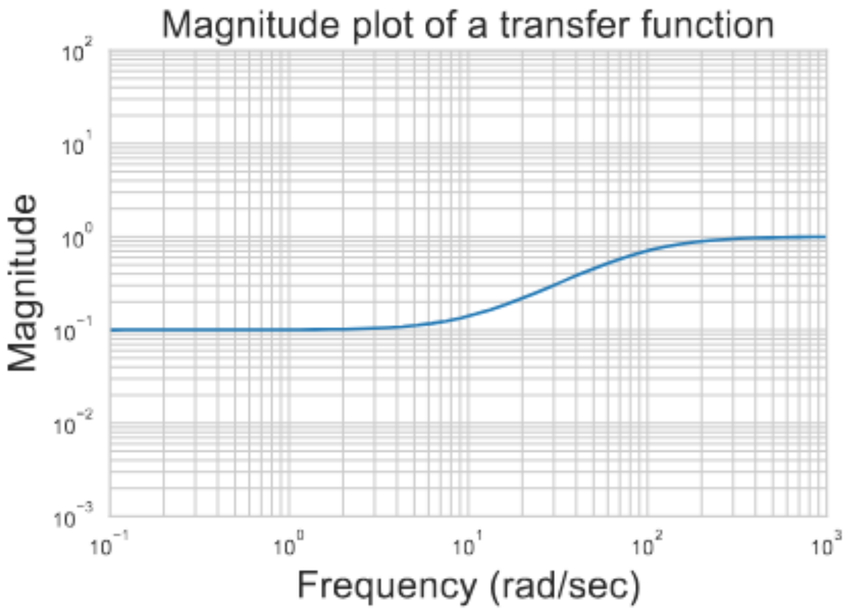
A)

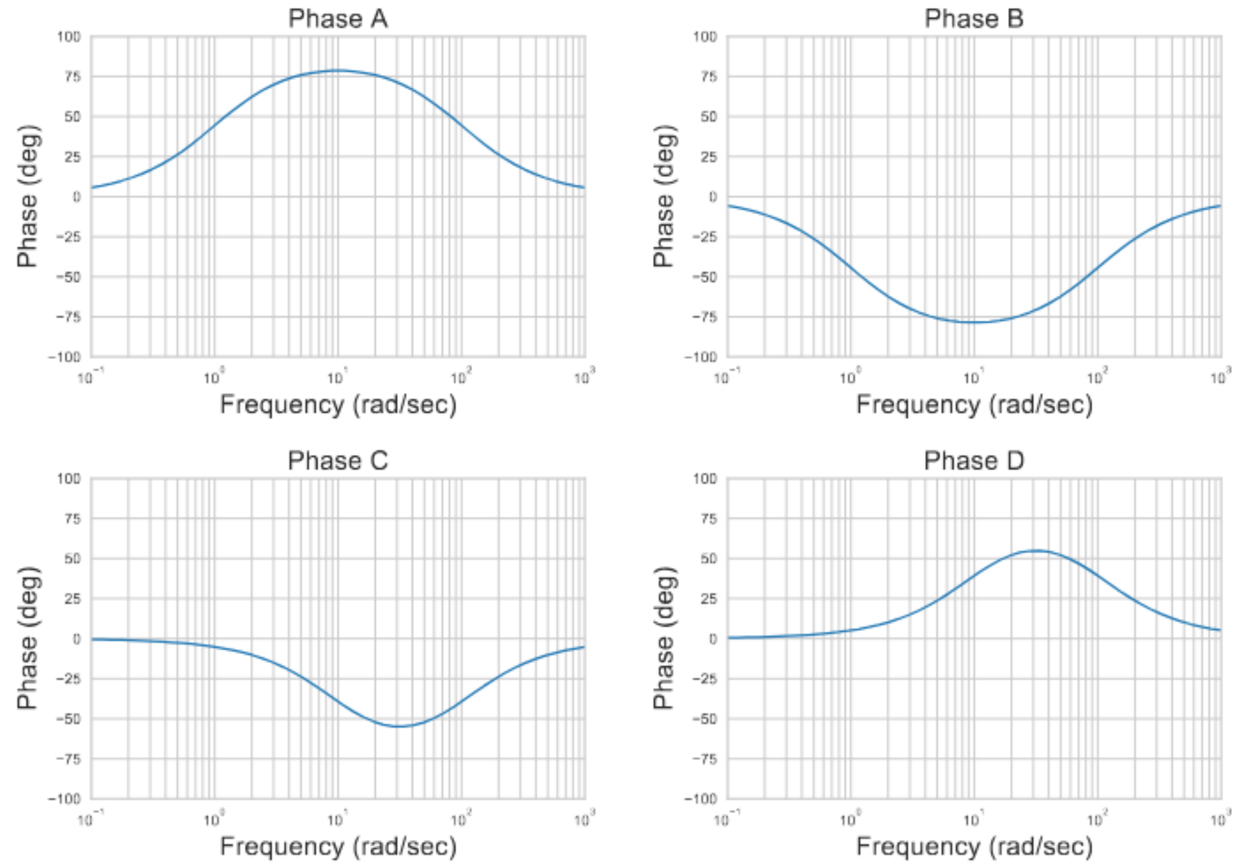
C)

B)

D)

FS 2018





Which of the following phase plots corresponds to the given magnitude plot?

A)

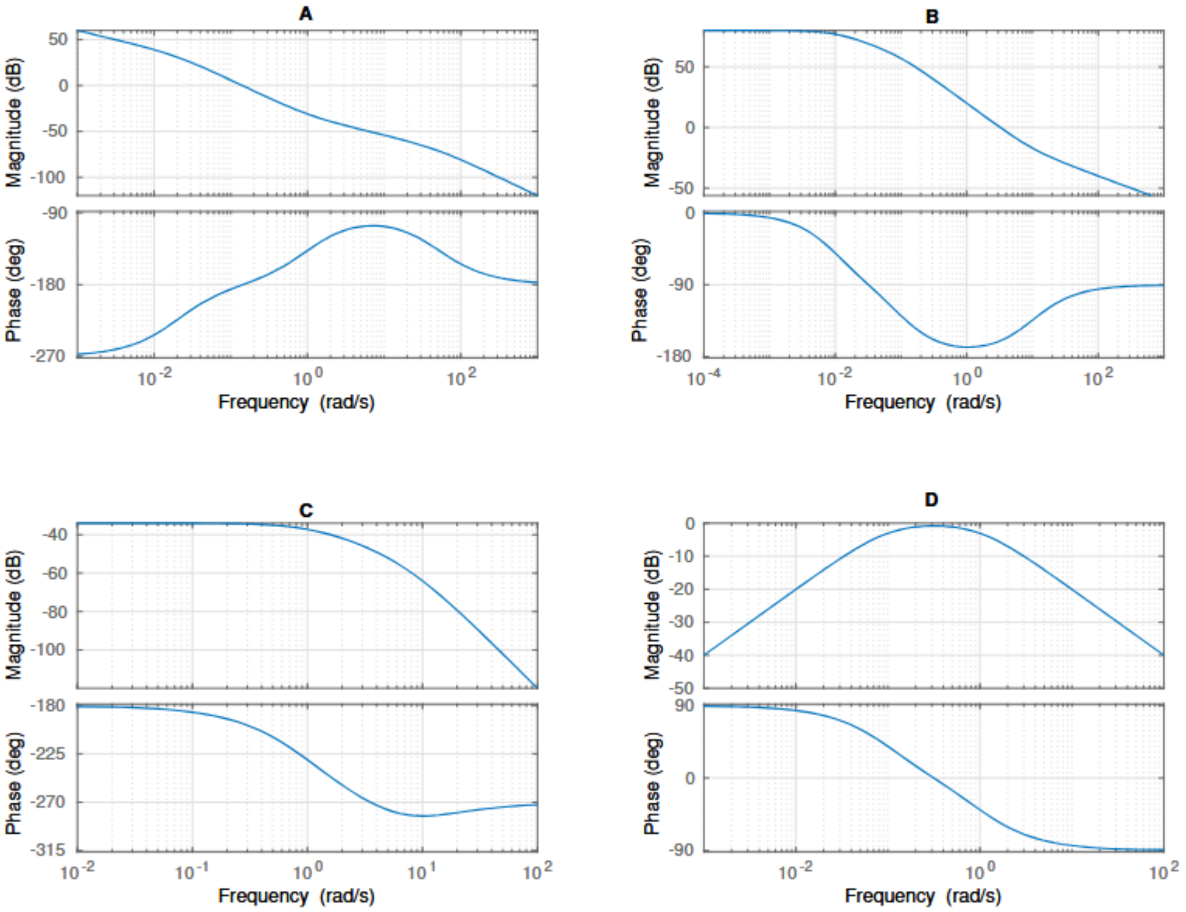
B)

C)

D)

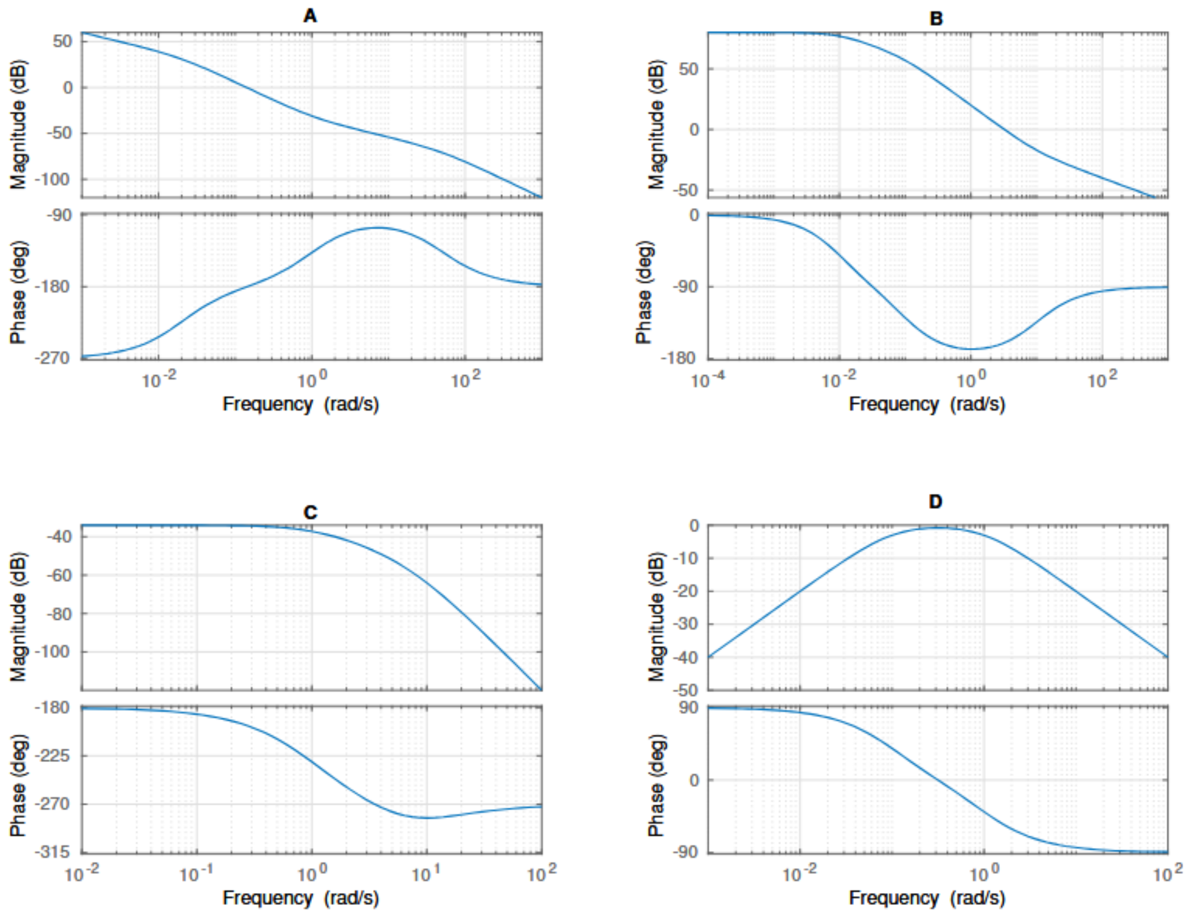
HS 2022

For you to look at if you want. Won't go through it now. However it is nice!



Transfer Function	A	B	C	D
$L_1(s) = \frac{s+1}{s(s+50)(s-0.02)}$				
$L_2(s) = \frac{s}{(s+1)(s+0.01)}$				
$L_3(s) = \frac{s+10}{(s+0.01)(s+0.1)}$				
$L_4(s) = \frac{1}{(s+1)(s-10)(s+5)}$				

HS 2022



Transfer Function	A	B	C	D
$L_1(s) = \dfrac{s+1}{s(s+50)(s-0.02)}$	x			
$L_2(s) = \dfrac{s}{(s+1)(s+0.01)}$				x
$L_3(s) = \dfrac{s+10}{(s+0.01)(s+0.1)}$		x		
$L_4(s) = \dfrac{1}{(s+1)(s-10)(s+5)}$			x	

Theory Recap

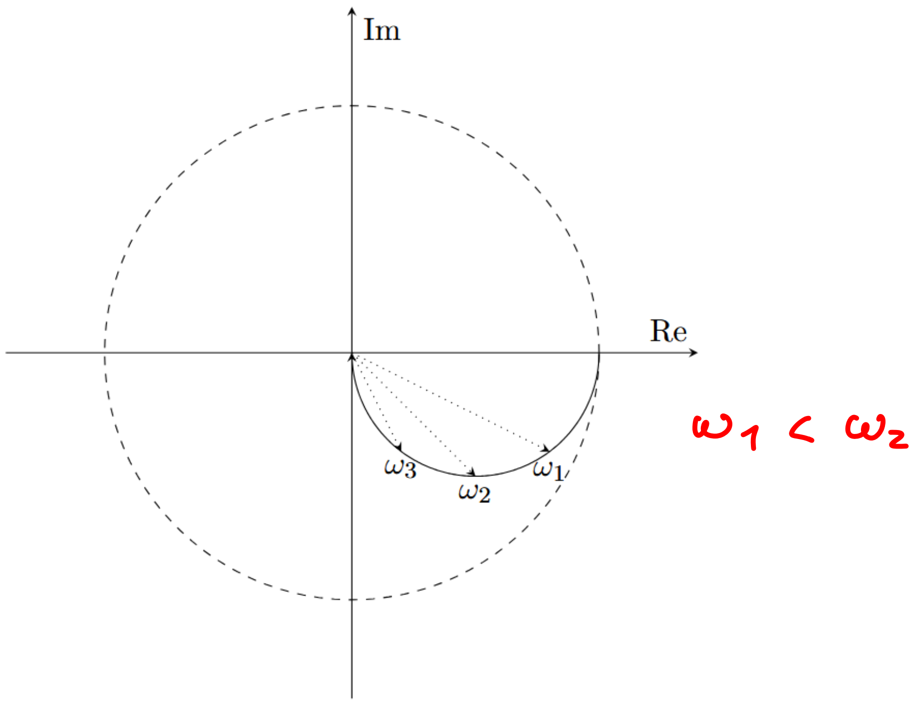
Polar Plot

Polar Plot

Remember how the Bode plot consisted of two subplots. One for the magnitude, and one for the phase.

For the Polar plot this is not the case. A single parametric curve in the complex plane with ω being implicit. Every point on the curve has a certain magnitude and a certain angle (phase), meaning that for every specific ω we get the magnitude and phase of our TF. (Same information as in Bode)

The Polar plot can be easily constructed by the Bode plot, since both show similar information



Polar Plot Construction

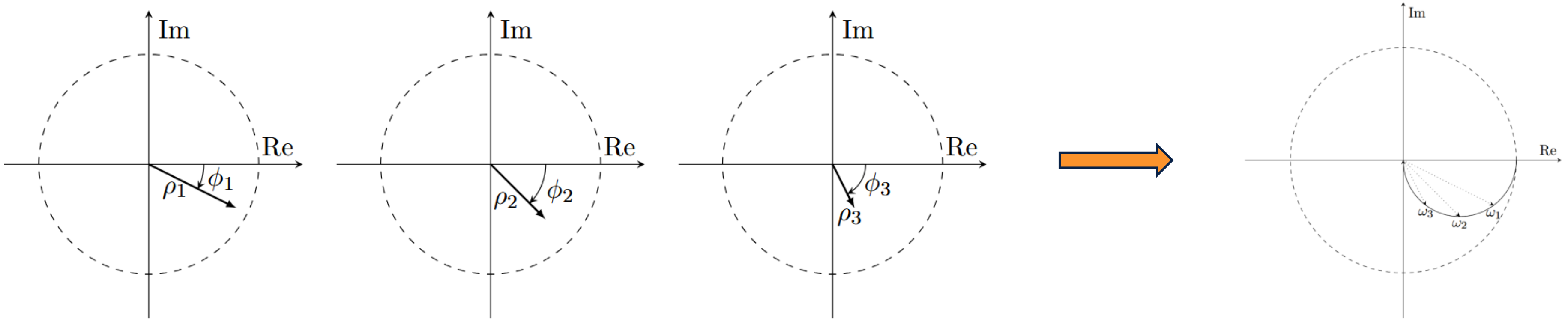
$$P(s) = \frac{1}{s+1}$$

Plugging in $s = j\omega$ and defining the **magnitude as ρ** and the **phase as ϕ** we look at different frequencies. In the end we get a continous line as on the plot from the slide before.

$$\omega_1 = 0.5, \quad \Rightarrow \rho_1 \approx 0.9, \quad \Rightarrow \phi_1 \approx -26.6°$$

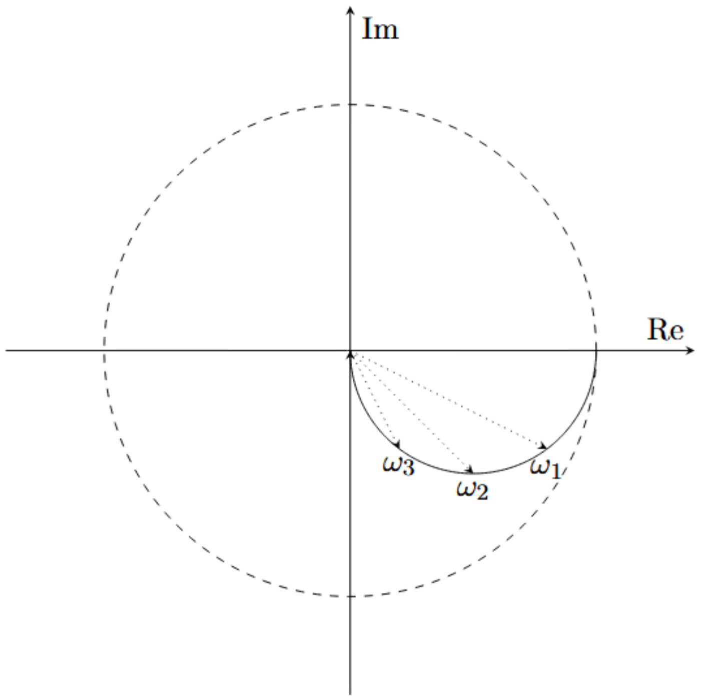
$$\omega_1 = 1, \quad \Rightarrow \rho_2 \approx 0.7, \quad \Rightarrow \phi_1 = -45°$$

$$\omega_1 = 2, \quad \Rightarrow \rho_1 \approx 0.4, \quad \Rightarrow \phi_1 \approx -63.4°$$

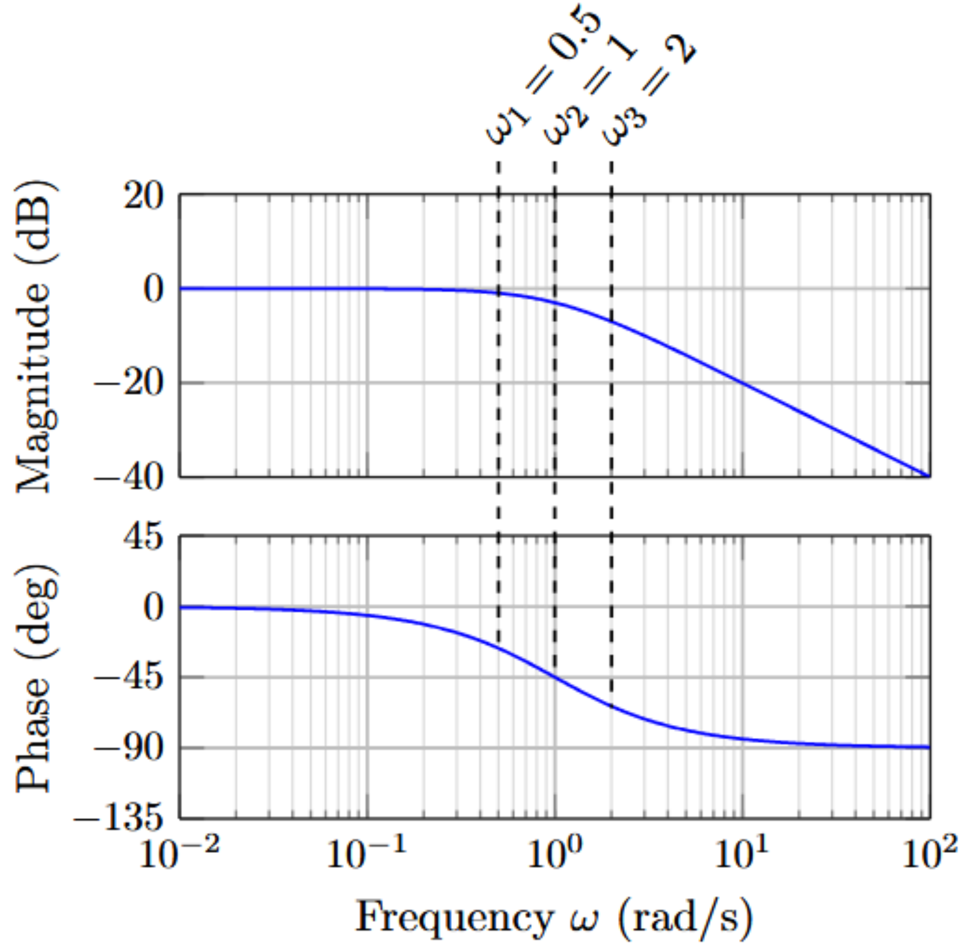


Remember from last week, how we determined that for a single stable pole and $\omega \to \infty$ the **phase drops by $-90°$** and that the **magnitude tends to $-\infty$ dB** (so 0 mangitude in decimals).
This behaviour can also be seen on the Polar plot.

Polar Plot – Bode Plot



=



Polar Plot Endpoint

Let $G(s) = \dfrac{N(s)}{D(s)}$ with $\deg N = n$ and $\deg D = m$. As $\omega \to \infty$,

$$G(j\omega) \sim \frac{c_n (j\omega)^n}{a_m (j\omega)^m} = \frac{c_n}{a_m}(j\omega)^{n-m}$$

so

$$|G(j\omega)| \sim \left|\frac{c_n}{a_m}\right| \omega^{n-m} \qquad \angle G(j\omega) \sim (n-m)\cdot 90^\circ + \angle\left(\frac{c_n}{a_m}\right)$$

$n = m \Rightarrow |G(j\omega)| = \text{const}$

This tells you whether the Polar curve ends at the **origin**, at a **nonzero finite point**, or at **infinity**, and from **which angle** it approaches.

$m > n \Rightarrow \frac{1}{\omega} \cdot \text{const}$

Nyquist Plot

Nyquist Plot

At first, we will only look at the Nyquist Plot and **how to construct** it.

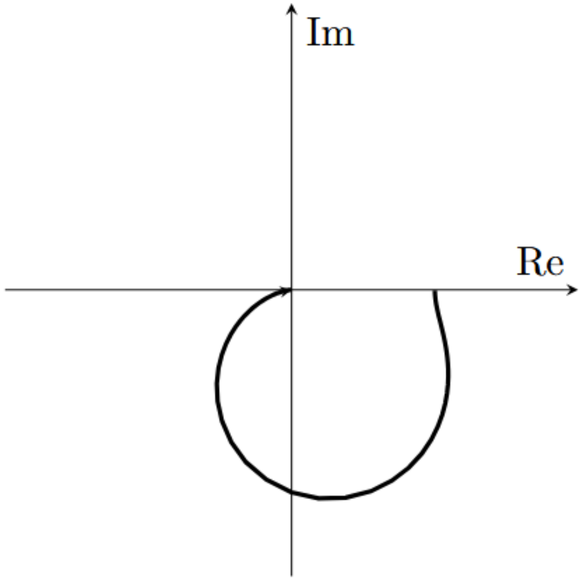
Later we will discuss what it tells us and how we can interpret it

Nyquist Plot Construction

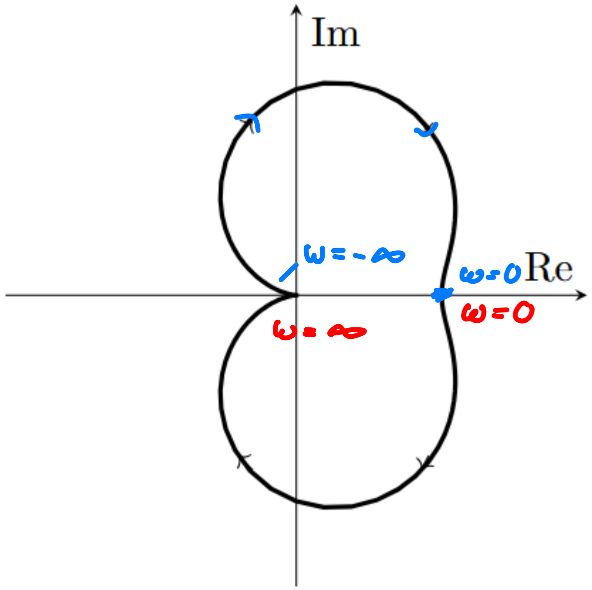
The **Polar plot** is a parametic curve in the complex plane for ω: $0 \rightarrow \infty$
The **Nyquist plot** is nothing but the completion of the Idea, meaning ω: $-\infty \rightarrow \infty$

Simplistically stated, it is just the Polar plot mirrored on the real axis, but continuing the direction of the parametric curve.

Polar Plot



Nyquist Plot



Nyquist Plot Construction

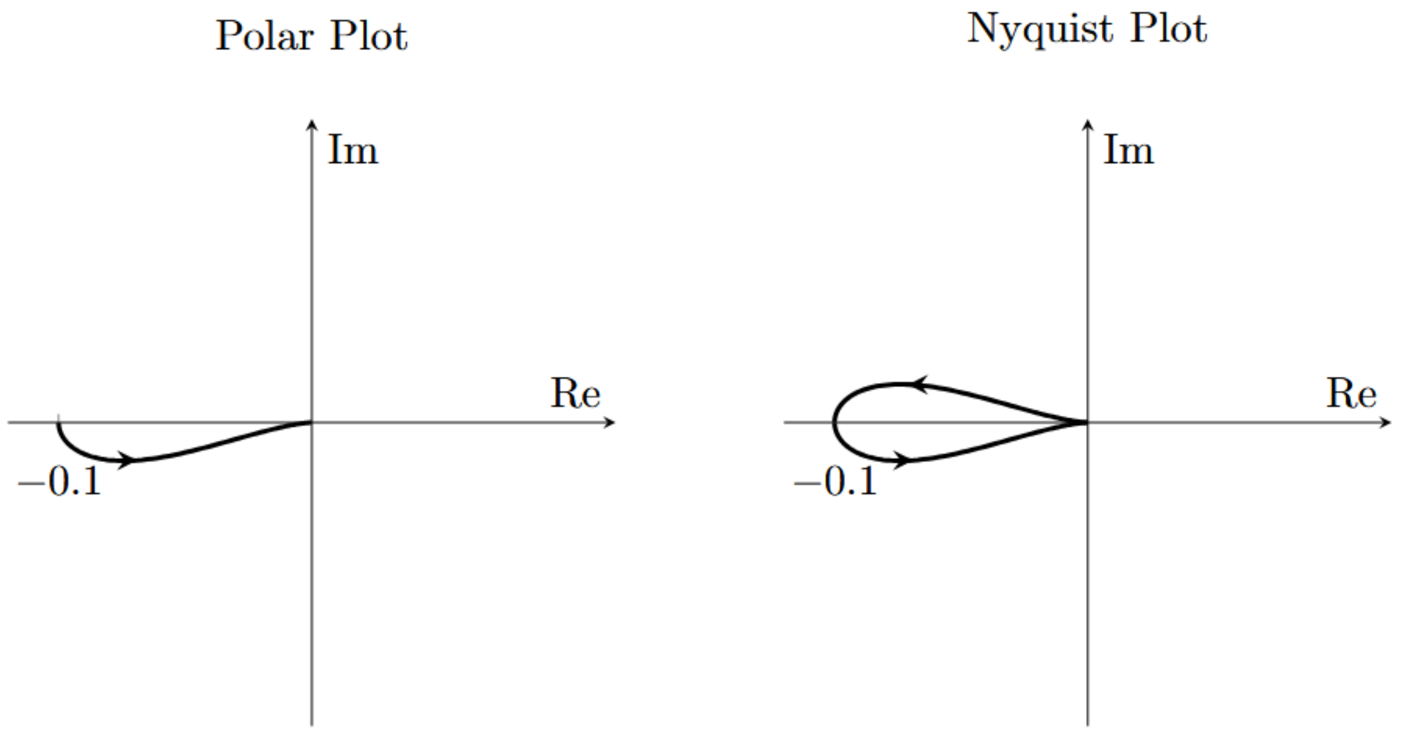
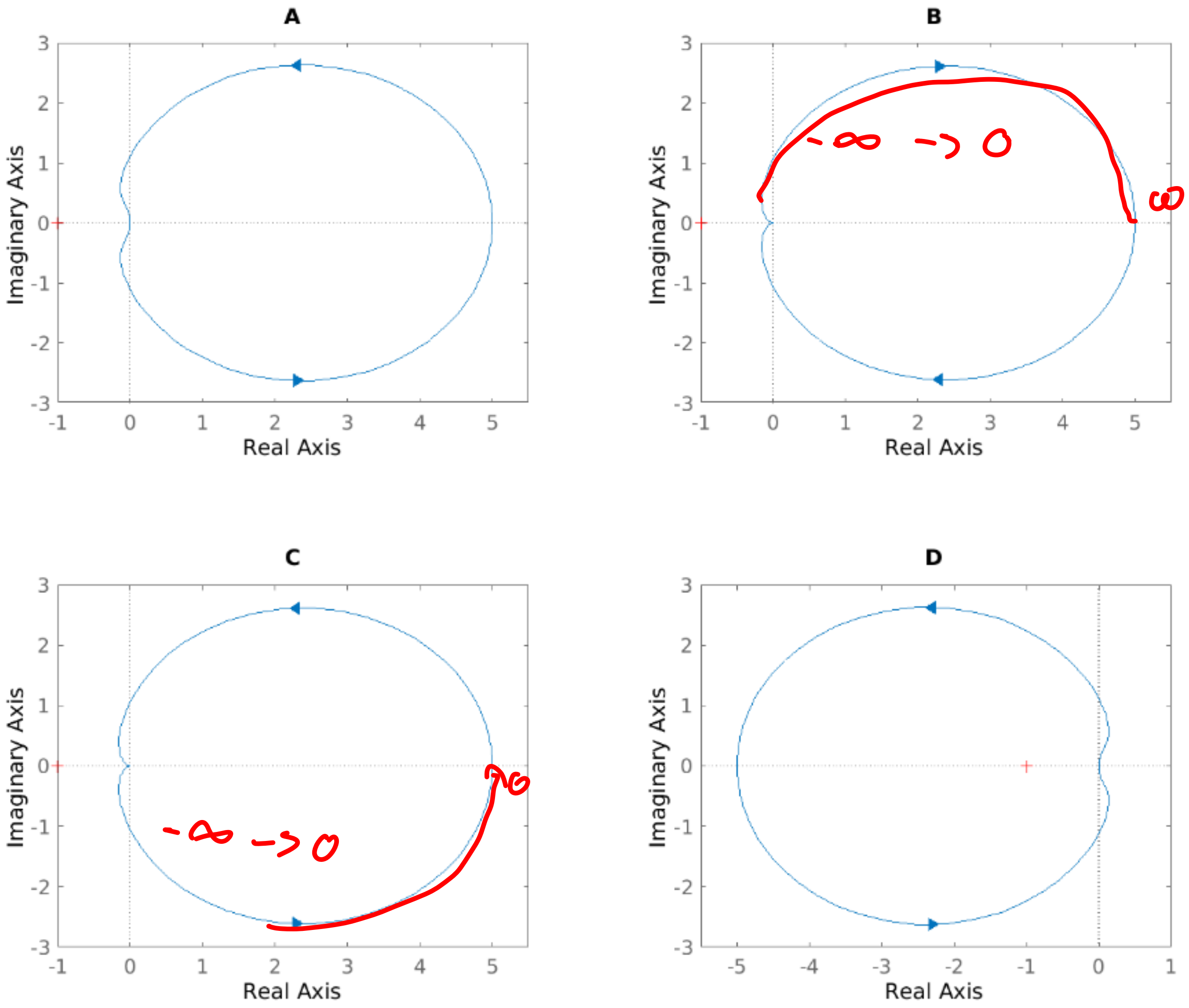


Figure 6.32: Polar and Nyquist plots of $L(s) = \frac{1}{s^2+s-10}$.

HS 2022

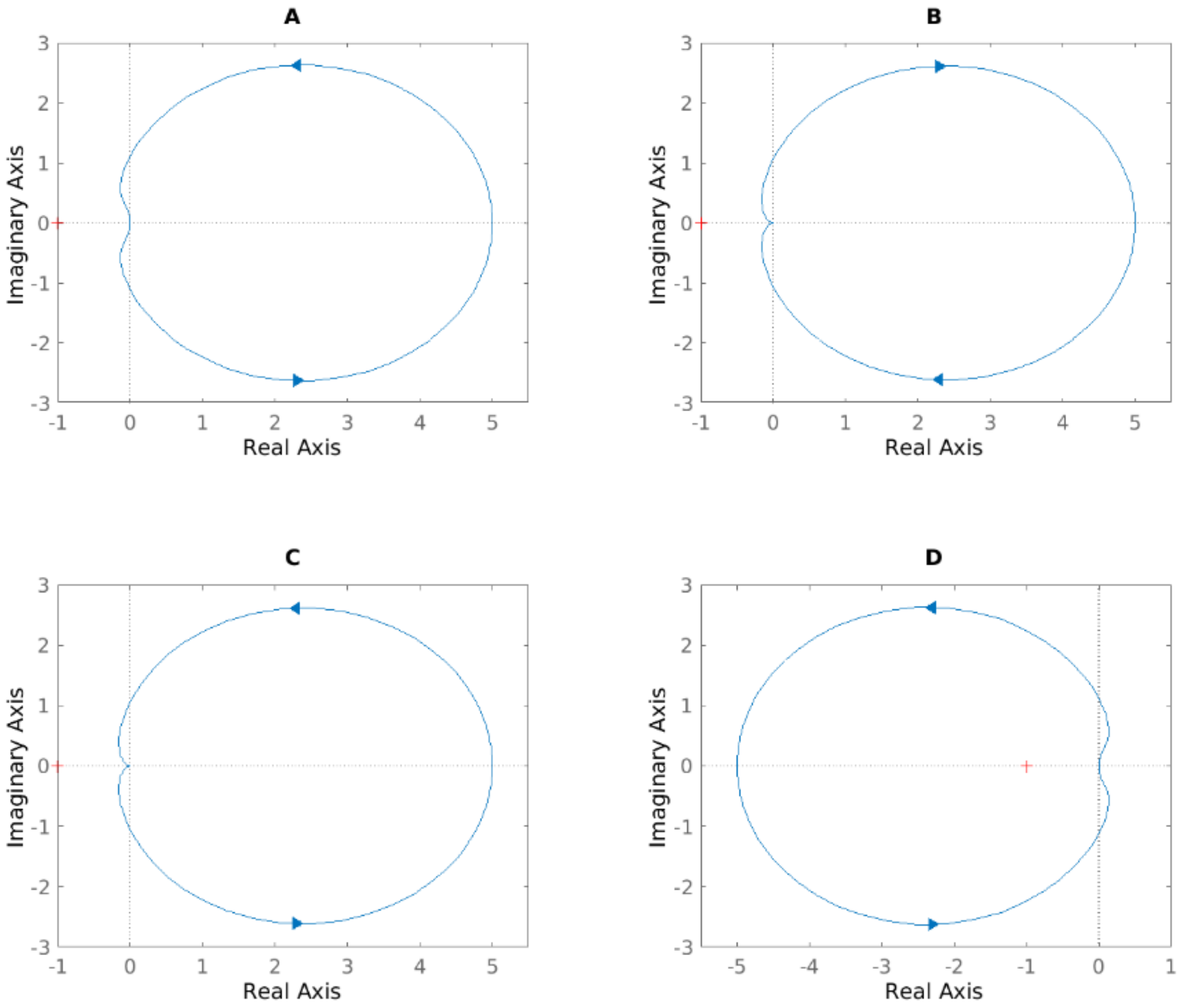
Phase \ Magnitude	−20 dB/dec	+20 dB/dec
−90°	stable pole	non-minimum phase zero
+90°	unstable pole	minimum phase zero



Transfer Function	A	B	C	D
$L_1(s) = \frac{-(s+1)}{(s+2)(s-0.1)}$				
≈ $L_2(s) = \frac{s+1}{(s+2)(s-0.1)}$				
$L_3(s) = \frac{1}{(s+2)(s+0.1)}$		X		
≈ $L_4(s) = \frac{1}{(s-2)(s-0.1)}$			X	

$L_3(0) = \frac{1}{2 \cdot 0.1} = \frac{1}{0.2} = 5$

HS 2022



Transfer Function	A	B	C	D
$L_1(s) = \frac{-(s+1)}{(s+2)(s-0.1)}$	x			
$L_2(s) = \frac{s+1}{(s+2)(s-0.1)}$				x
$L_3(s) = \frac{1}{(s+2)(s+0.1)}$		x		
$L_4(s) = \frac{1}{(s-2)(s-0.1)}$			x	

Why Nyquist

We will make a quick detour to complex analysis, but you will see why we investigated this plot!
But as a teaser:

With Nyquist, we will always be able to assess the closed loop stability, always!
No matter how the open loop system looks like

Nyquist Theorem

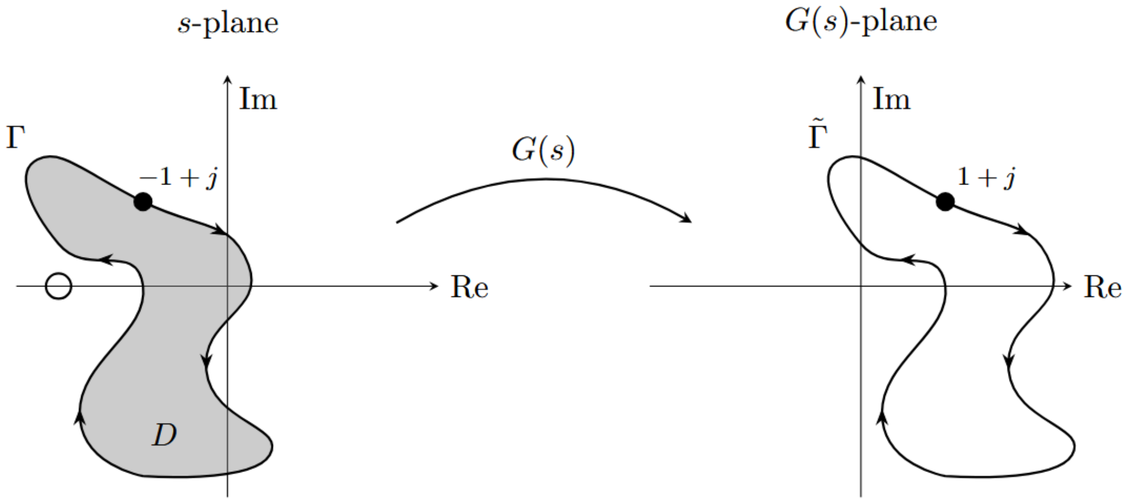
Motivation

The next steps may seem random at first, but try to just follow the individual steps and in the end we will **make a connection** between them and **extract the sense** behind it.

Principle of the Variation of the Argument

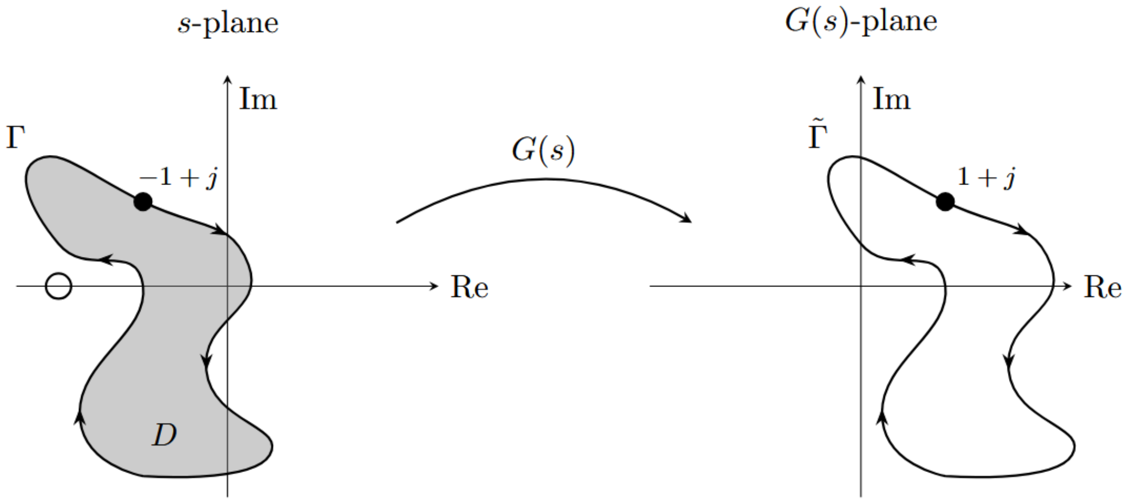
$$G(s) = s + 2$$

- Let us consider the transfer function from above and think about what it does to a point.
  → It takes the point and shifts it to the right, by adding + 2.
- Generally, G(s) takes any complex point from the **s - plane** and **maps it to a new plane**, we call it **G(s) - plane**.
- Looking at the s - plane, as always we will depict G(s) by indicating the position of the zeros and poles.



Principle of the Variation of the Argument

- We can of course not only map only certain points, but we can map entire continuous curves.
- Particularly interesting are closed curves, that describe a bounded region D.
- We then define the **contour Γ**, obtained by following the boundary of D in **clockwise (CW)** direction:



Principle of the Variation of the Argument

As always we can look at the phase and magnitude of the TF seperately. For now we are particularly interested what the TF does to any point phasewise. For that we consider the following TF, that we can also write generally:

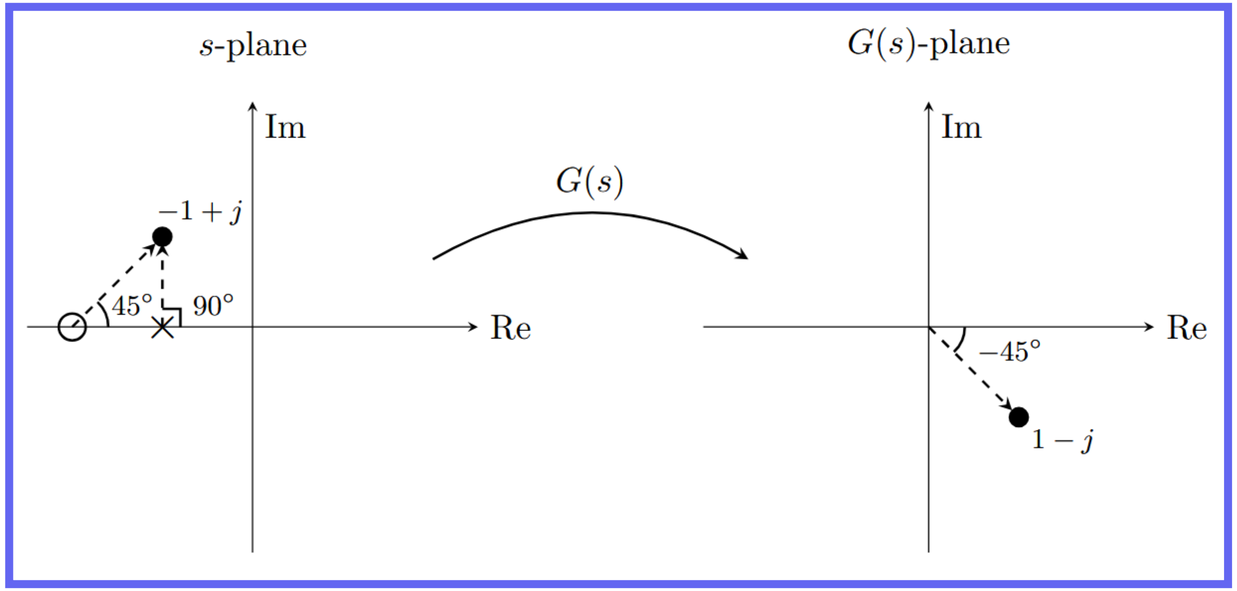
$$G(s) = \frac{s+2}{s+1} = \frac{s-z_1}{s-p_1}$$

Now remember how we compute the phase for any point

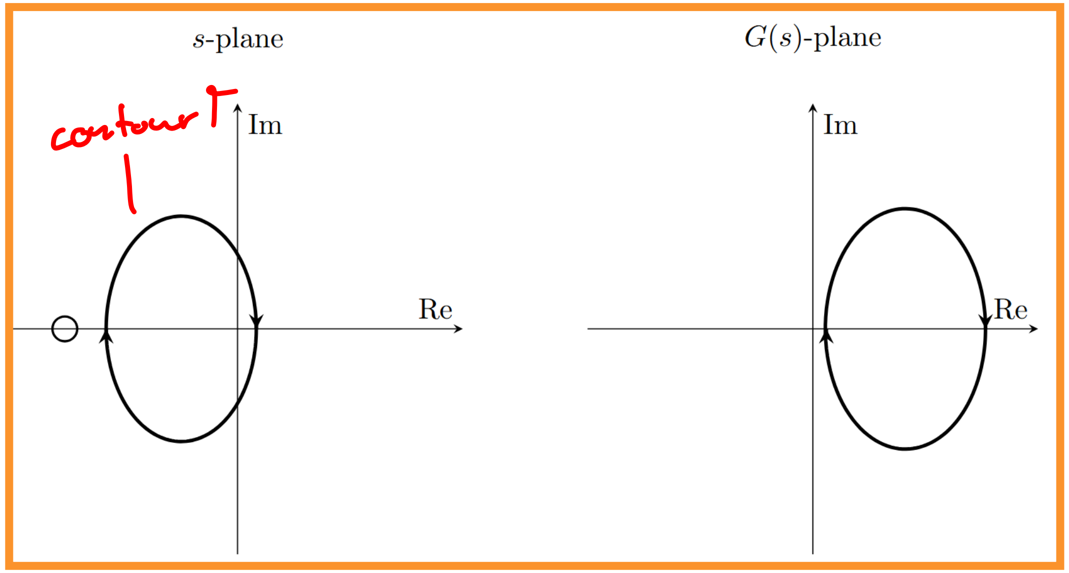
$$\angle(G(s_0)) = \angle\left(\frac{s_0-z_1}{s_0-p_1}\right) = \angle(s_0-z_1) - \angle(s_0-p_1)$$

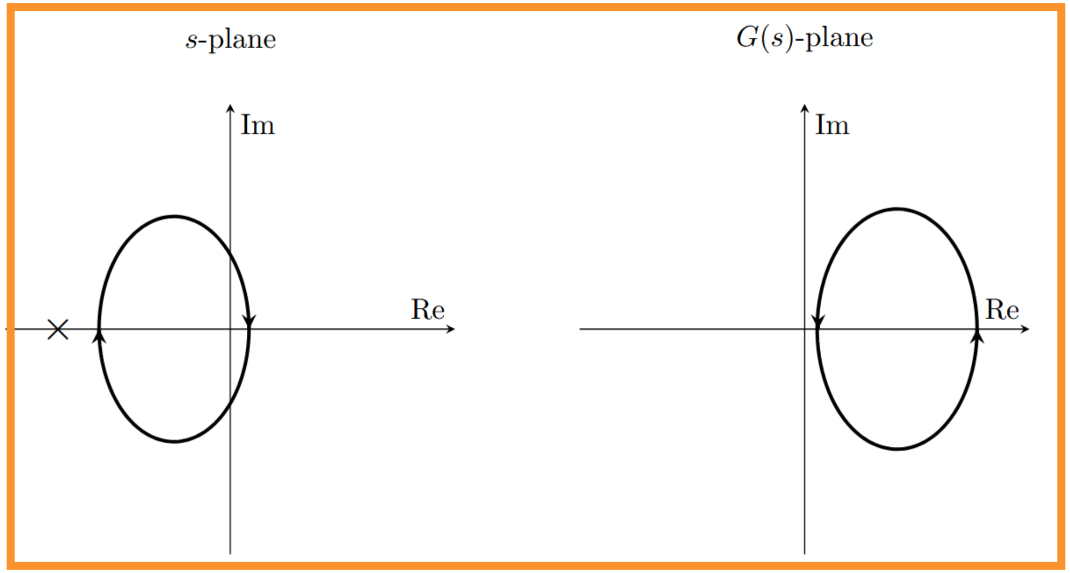
General rule therefore for the G(s) - plane:

Add phase of zeros, subtract phase of poles

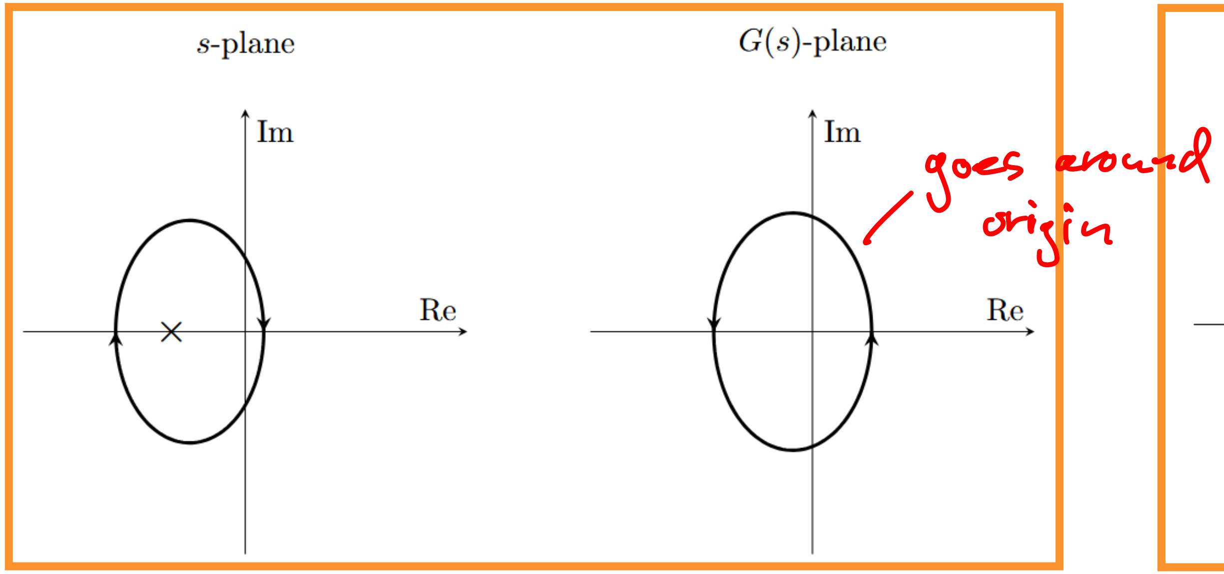


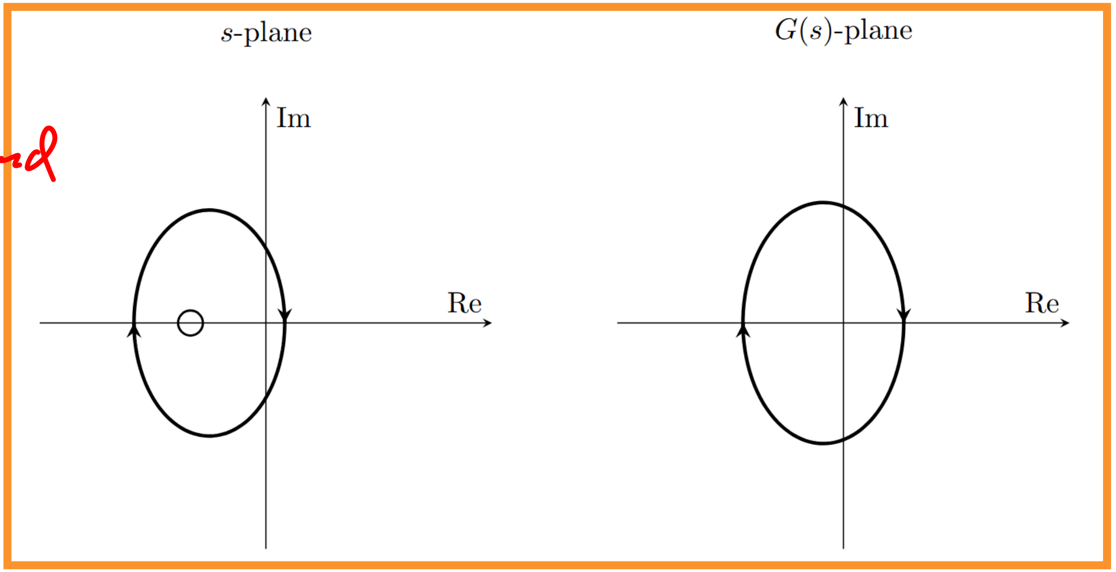
Examples

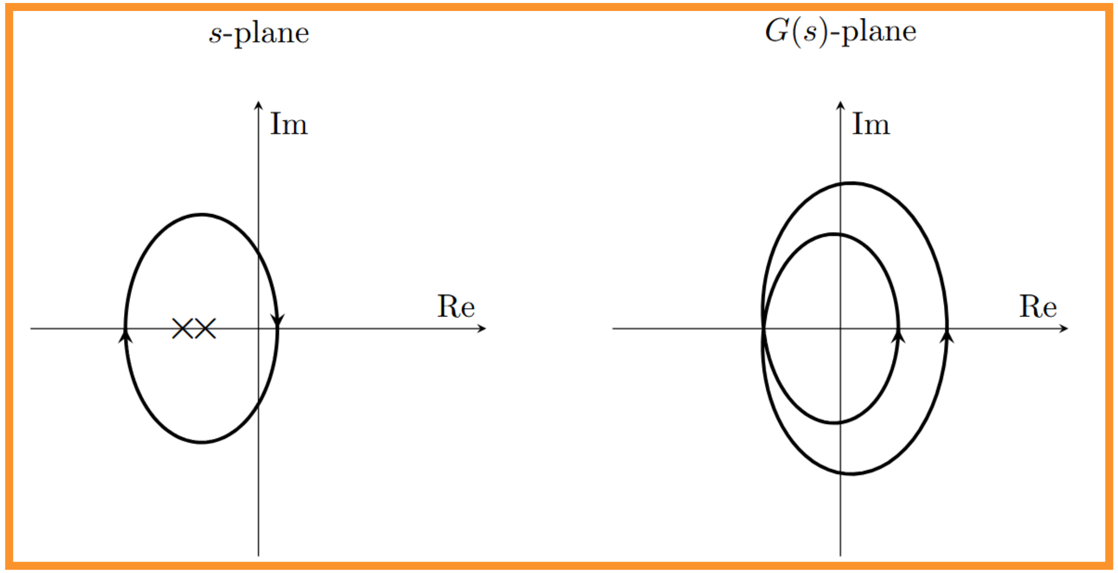




More Examples

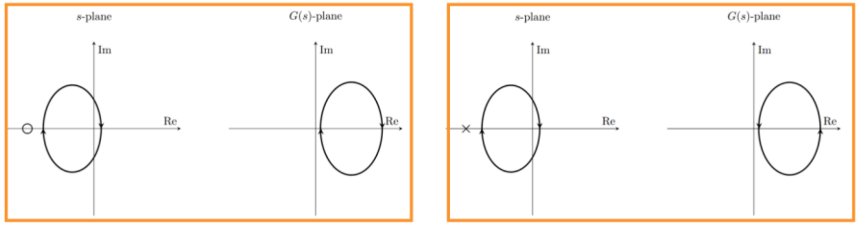


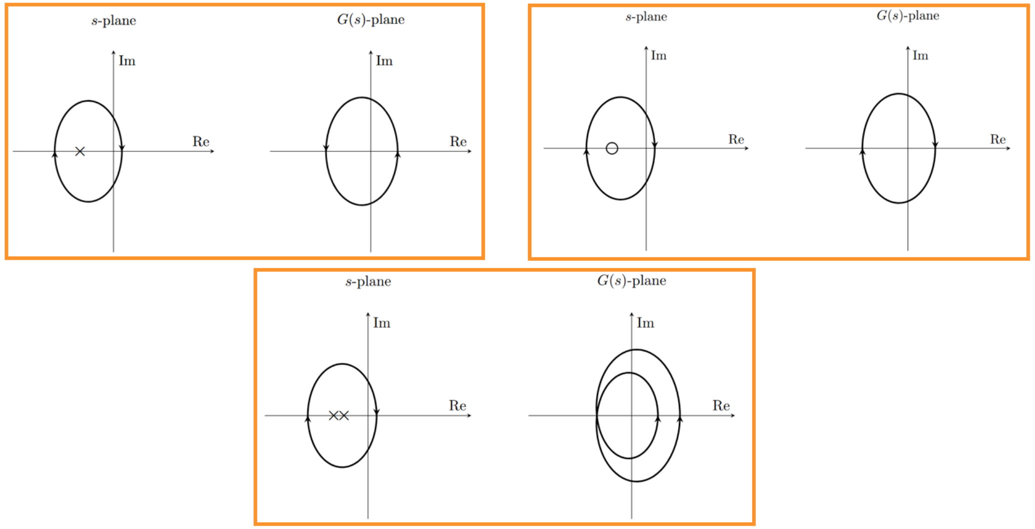




Principle of the Variation of the Argument

We can see that for **every time we encircle a pole or zero in the s - plane we also encircle The origin in the G(s) - plane**. For each CW encirclement of zeros we get one CW encirclement of the origin and for each CW encirclement of a pole we get one CCW encirclement of the origin. **Think of a zero adding 360°and a pole subtracting 360°.**





Principle of the Variation of the Argument

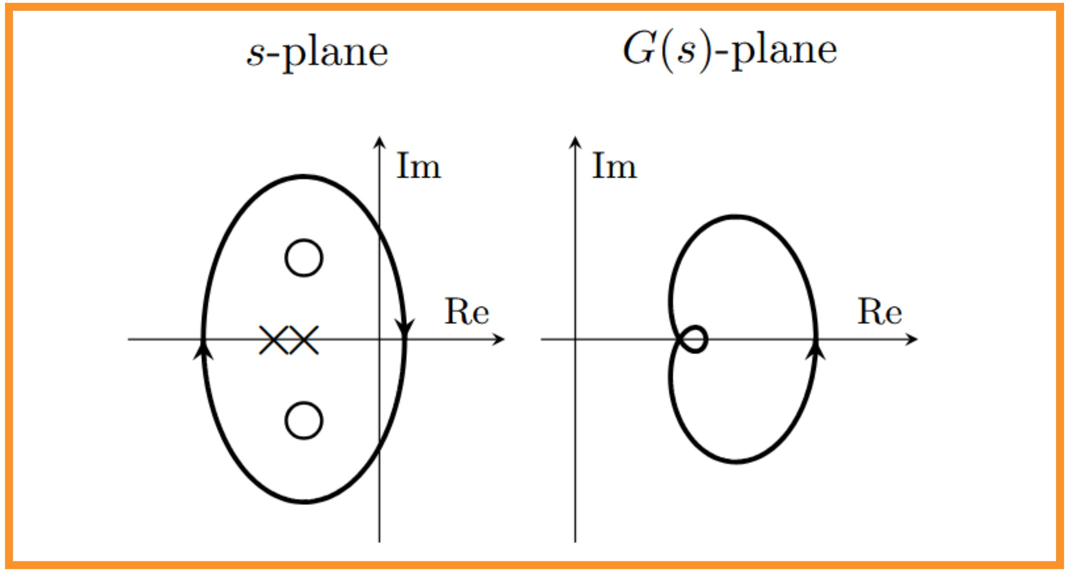
Theorem (Variation of the argument [Proof in A&M, pp. 277–278])

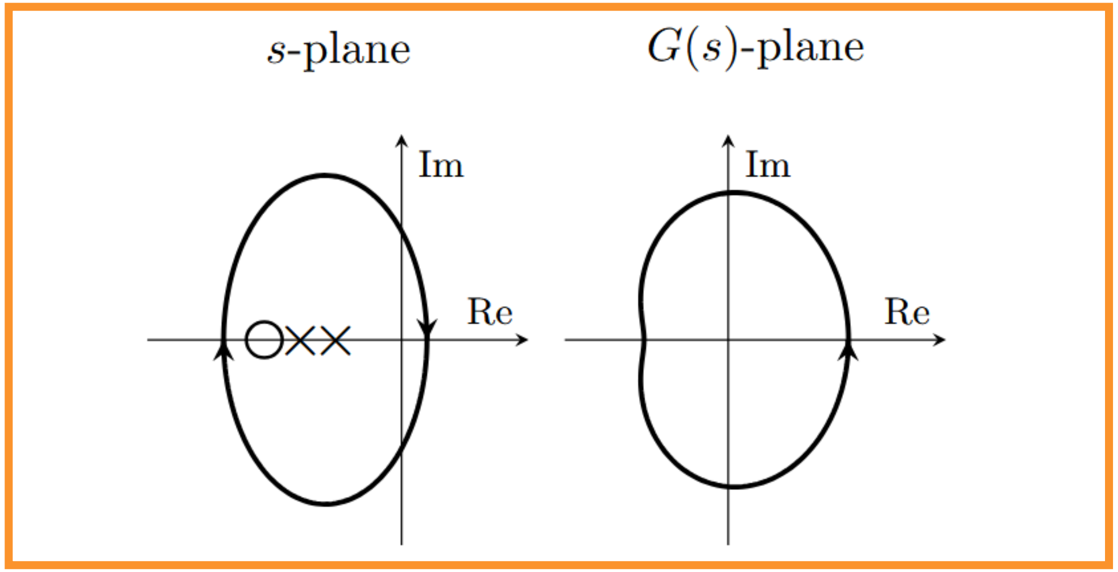
The number N of times that $G(s)$ encircles the origin of the complex plane as s moves along the boundary Γ of a bounded simply-connected region of the plane satisfies

$$N = Z - P,$$

where Z and P are the numbers of zeros and poles of $G(s)$ in D, respectively. Note that the encirclements are counted positive if in the same direction as s moves along Γ, and negative otherwise.

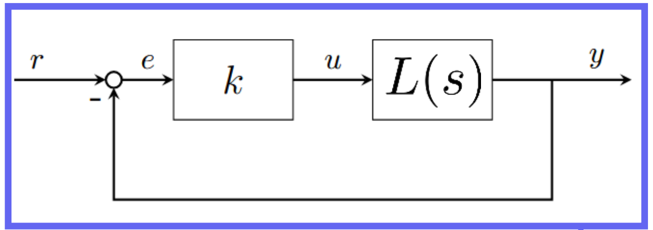
More More Examples





Nyquist Stability

Why do we do this?



- Generally we want to asses the stability of a given system / transfer function. For an open loop TF $L(s)$ we have to check if there are any poles in the RHP

- For the closed loop system with some feedback k, $T(s) = \frac{kL(s)}{1+kL(s)}$ we have to do the same.

- Here this means checking $\mathbf{1 + kL(s) = 0}$

- So generally **assessing the stability of $T(s)$ means checking for the zeros of $1 + kL(s)$**

Nyquist Contour

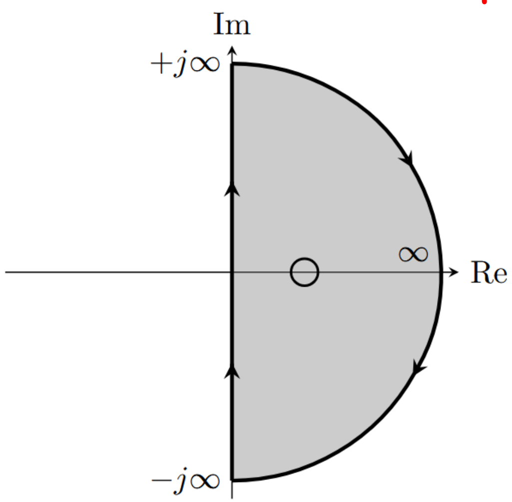
This matches with our intuition for the Nyquist plot, where we said we let **s** go from $-j\infty \rightarrow j\infty$. So we obtain the Nyquist plot by letting s change along the Nyquist contour.

Now we connect with the variational Princpile!

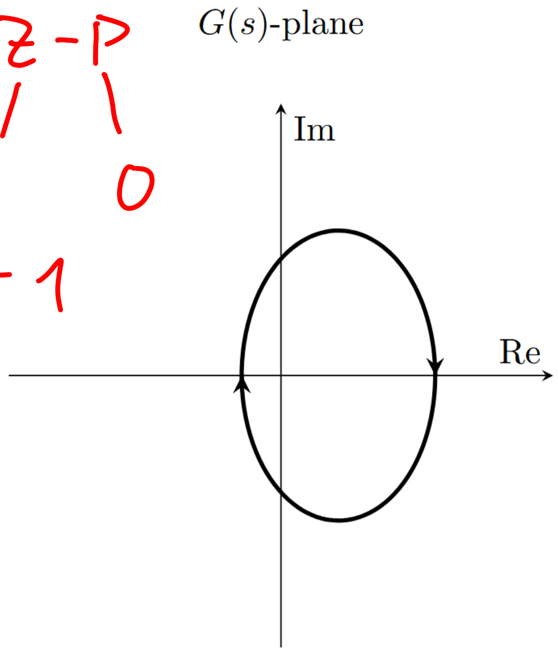
Let us introduce our own contour, the so called **Nyquist contour**. The important thing is, that it includes the entire right half-plane (RHP) by going from $-j\infty \rightarrow j\infty$, and closing in a **semicircle at** ∞

Now the entire RHP, with all its zeros and poles, is bounded by our contour.
~~Let us choose~~

The origin encirclements of the Nyquist plot of $1 + kL(s)$ tells us about the relative amount of zeros and poles.



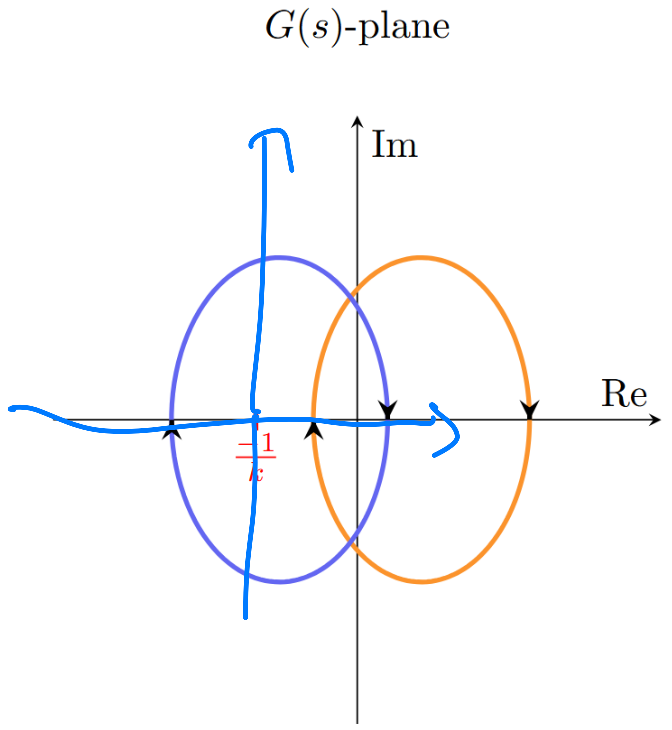
$N = Z - P$
1 0
$N = 1$



Shifting Coordinate System

So we said we can look at the Nyquist plot of $1 + kL(s)$ to assess the relative degree.
But usually we only know the open loop TF $L(s)$ and ist Nyquist Plot.

→ Luckily, we can just shift the coordinate system, meaning we can now count the $L(s)$ encirclements of the point $-\frac{1}{k}$



Meaning…

Enc. of origin (of Nyquist plot) of $1 + kL(s)$ = # RHP zeros of $1 + kL(s)$ − RHP poles of $1 + kL(s)$

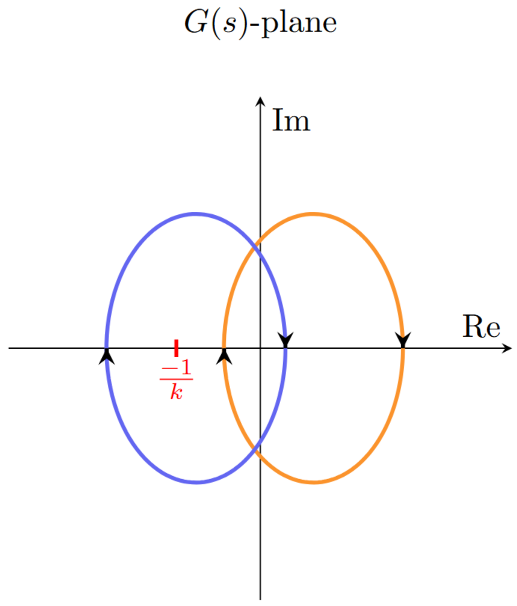
Enc. of origin of $1 + kL(s)$ = # Enc. of $\frac{-1}{k}$ of $L(s)$

Lets quickly look at the poles of $1 + kL(s)$:

Remember: $L(s) = \frac{N(s)}{D(s)}$, $1 + kL(s) = 1 + \frac{kN(s)}{D(s)} = \frac{D(s) + kN(s)}{D(s)}$

➞ We can see that **they both share the same poles**, that are equal to **the open loop unstable poles**!

Enc. of $\frac{-1}{k}$ of $L(s)$ = # RHP zeros of $1 + kL(s)$ − RHP poles of $L(s)$



All coming together!!

Enc. of $\frac{-1}{k}$ of $L(s)$ = # RHP zeros of $1 + kL(s)$ − RHP poles of $L(s)$

Now recall that we said want to find the unstable poles of the closed loop TF: $T(s) = \frac{kL(s)}{1+kL(s)}$

And this is nothing else but finding the RHP zeros of $1 + kL(s)$.

This all leads us to the following formula N = Z – P, or reformulated:

$$Z = N + P$$

Z = # Unstable Closed-Loop Poles
N = # CW enc. of $L(s)$
P = # Unstable Open-Loop Poles

This is awesome because it allows us to assess the closed loop stability by only looking at the open loop system $L(s)$

HS 2022

Problem: Consider the closed-loop system T in Figure 14, where L is a linear time-invariant system with two unstable poles.

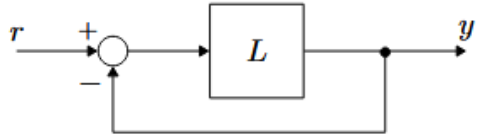


Figure 14: Closed-loop system T.

$k \neq 1 \Rightarrow k = 10$

$-\frac{1}{k} = -1, \quad k = 1$

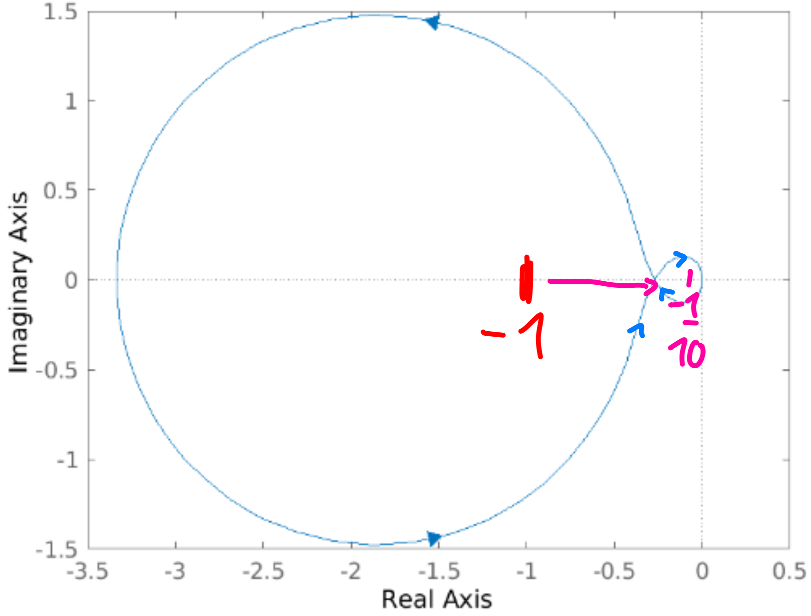


Figure 15: Nyquist plot of the transfer function of L.

$Z = 1 + 2 = 3$

The closed Loop System T is unstable

True

False

#enc. of $-\frac{1}{k}$ by $L(s)$

$N + P = Z = 1$

N: $0 = -1$, P: 2, Z: unstable CL-Poles

HS 2022

Problem: Consider the closed-loop system T in Figure 14, where L is a linear time-invariant system with two unstable poles.

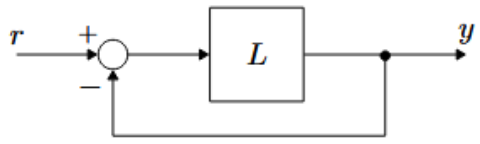


Figure 14: Closed-loop system T.

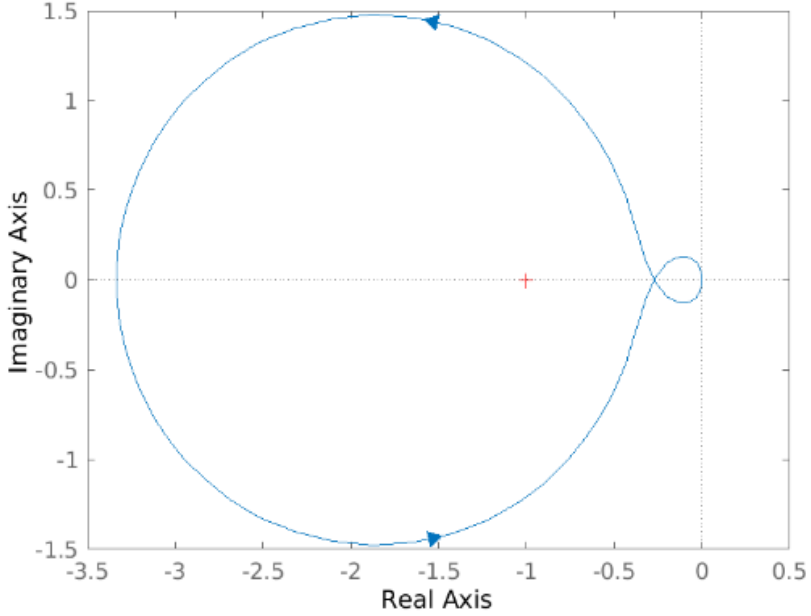


Figure 15: Nyquist plot of the transfer function of L.

The closed Loop System T is unstable

True

~~False~~

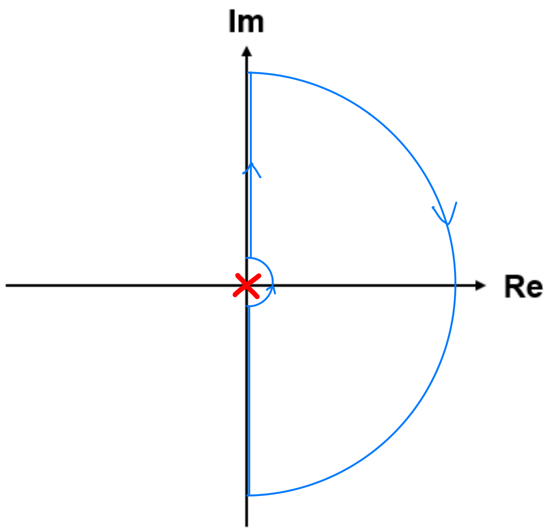
unstable

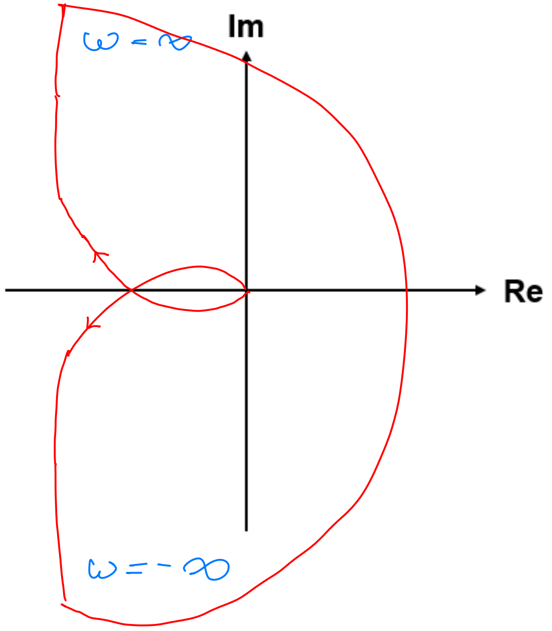
$$Z = N + P$$
$$Z = -1 + 2 = 1$$

Special Case

What happens when there are poles on the Im – axis of the open loop system? Do they count into the Contour or not?
For that we will just follow a special rule:

When excluding the poles, going around them CCW, close the Nyquist plot at infinity clockwise. For every pole excluded, add $+180°$ when closing the Nyquist





HS 2022

Which of the following Statements is True?

Statement
The Nyquist plot is symmetric about the real-axis.
When applying the Nyquist stability criterion, it does not matter whether encirclements of the critical point happen in clockwise or counter-clockwise direction.
The Nyquist stability criterion cannot be applied if the open-loop system has non-minimum phase zeros.

A)

B)

C)

HS 2022

Which of the following Statements is True?

Statement
The Nyquist plot is symmetric about the real-axis.
When applying the Nyquist stability criterion, it does not matter whether encirclements of the critical point happen in clockwise or counter-clockwise direction.
The Nyquist stability criterion cannot be applied if the open-loop system has non-minimum phase zeros.

A)

B)

C)

Statement 1: True. The Nyquist plot is created by mirroring the polar plot of $L(j\omega)$ across the real-axis, and is correspondingly always symmetric with respect to the real-axis.

Statement 2: False. The Nyquist criterion requires the net number of encirclements. When the Nyquist contour is traversed in a clockwise direction, clockwise encirclements ↻ are counted positive, and counter-clockwise ↺ are counted negative.

Statement 3: False. The Nyquist theorem can be used for systems with non-minimum phase zeros.

Frequency Domain Specifications

Stability Margins

Let us consider an open-loop stable system:
From the Nyquist criterion $Z = N + P$ we know that the Nyquist plot of $L(s)$ needs to encircle the -1 point (assumed $k = 1$) a **net amount of 0 times** (due to $P = 0$), for the closed-loop system to be stable.

We now can define **stability margins**, that tell us how close our system is to being unstable.

In other words, we want our $|L(j\omega)| < 1$ whenever $\angle L(j\omega) = \pm 180°$.

From that we can deduce 2 stability margins:

Gain Margin GM / $g_{m,dB}$: It indicates how much we can increase the magnitude at the phase crossover frequency ω_{pc}, at which the phase crosses $\pm 180°$, before encircling -1

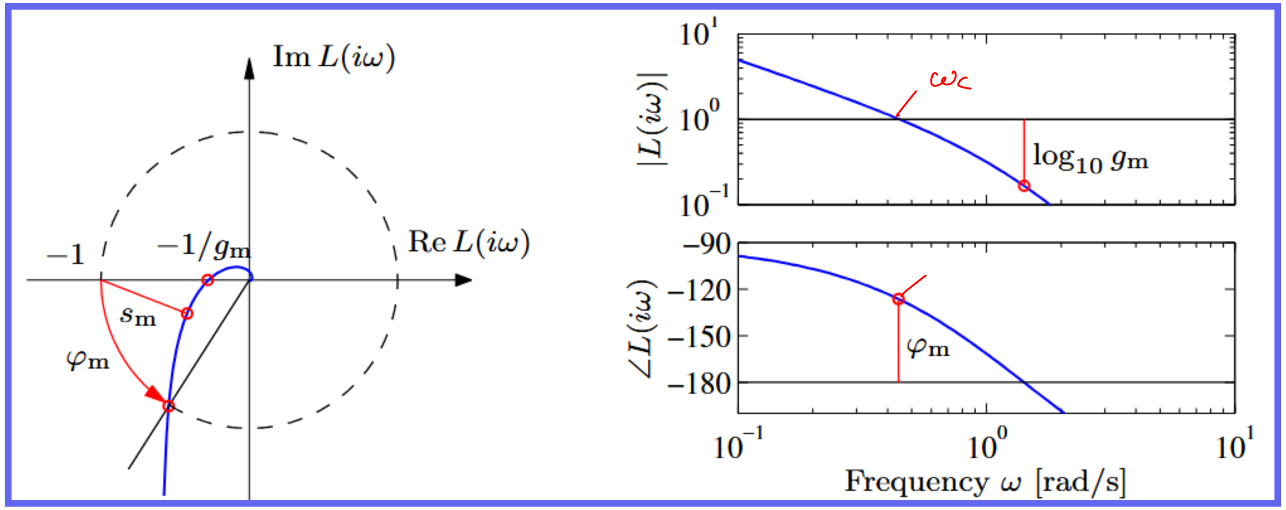
Phase Margin PM / φ_m: It indicates much much we can change the phase at the crossover frequency ω_c, at which the magnitude is 1 (0 in dB), before encircling -1

Stability Margins

From the previous we can deduce 2 stability margins:

Gain Margin GM / $g_{m,dB}$: It indicates how much we can increase the magnitude at the phase crossover frequency ω_{pc}, at which the phase crosses $\pm 180°$, before encircling -1

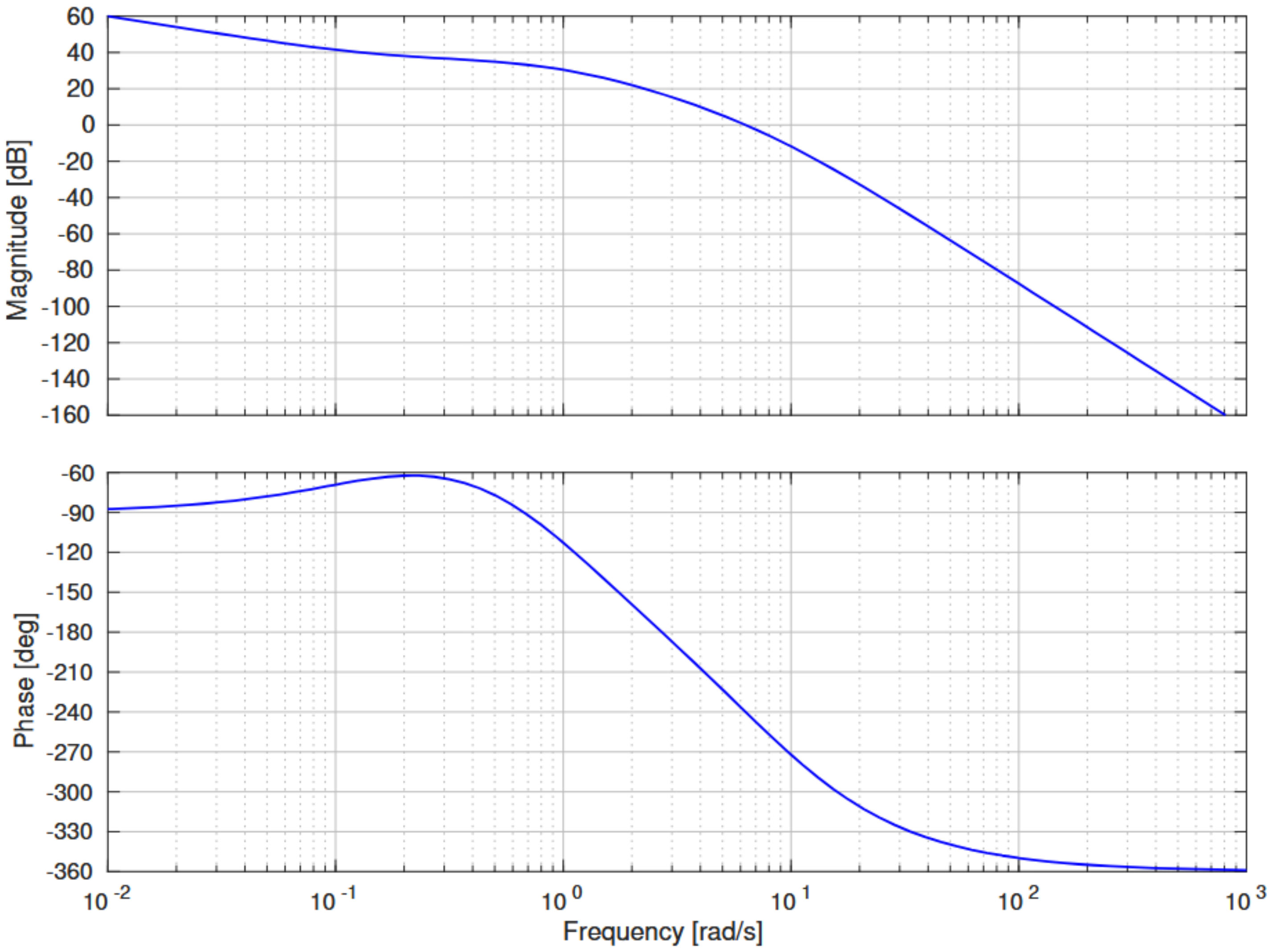
Phase Margin PM / φ_m: It indicates much much we can change the phase at the crossover frequency ω_c, at which the magnitude is 1 (0 in dB), before encircling -1



Phase margin:	Gain margin:
Find **cross-over frequency** ω_c: $\lvert L(j\omega_c)\rvert = 1$, $\lvert L(j\omega_c)\rvert_{dB} = 0$	Find the frequency ω at which $\angle L(j\omega) = -180°$. Then find $\lvert L(j\omega)\rvert_{dB}$ for the same ω
Find the phase $\varphi = \angle L(j\omega_c)$ and set the phase margin: $\varphi_m = \varphi + 180°$	Compute the gain margin: $g_{m,dB} = 0 - \lvert L(j\omega)\rvert_{dB}$, $g_m = 10^{\frac{g_{m,dB}}{20}}$

FS 2024

For you to look at if you want. We'll do it together if we have time…



The PM is best described by:

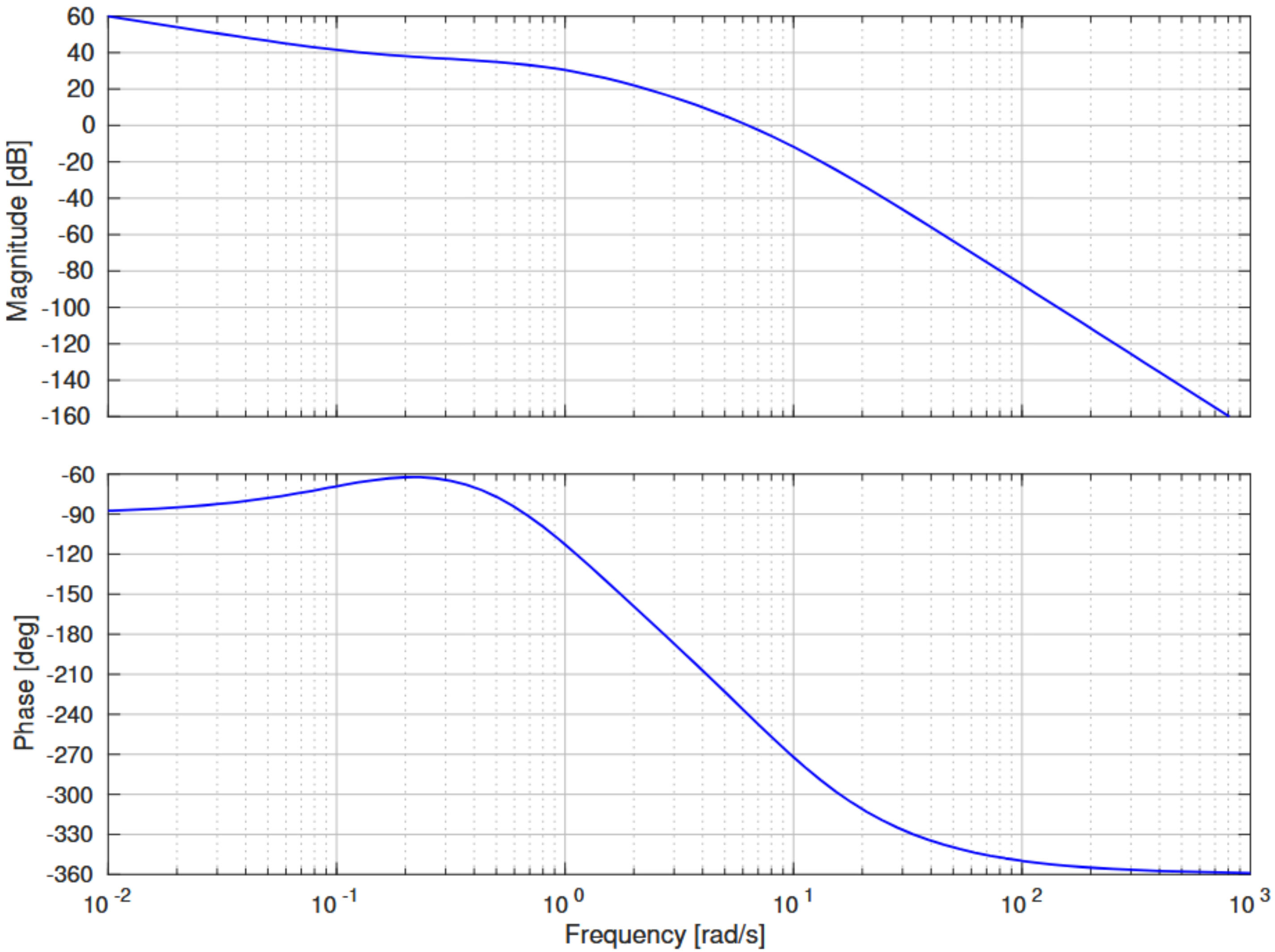
A) $PM \approx 58°$

B) $PM \approx -61°$

C) $PM \approx 34°$

D) $PM \approx -26°$

FS 2024



The PM is best described by:

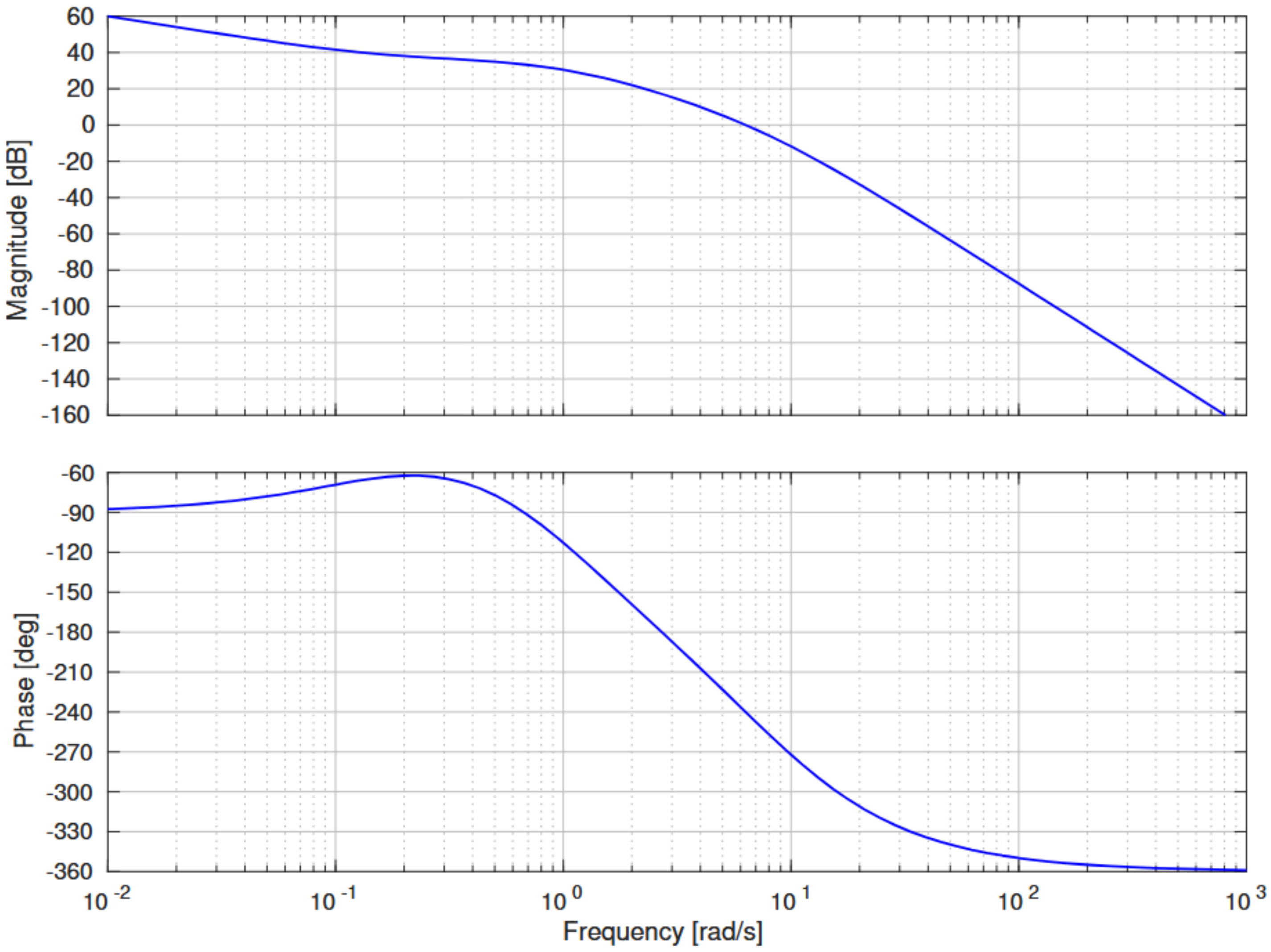
A) $PM \approx 58°$

B) $PM \approx -61°$

C) $PM \approx 34°$

D) $PM \approx -26°$

FS 2024



The GM is best described by:

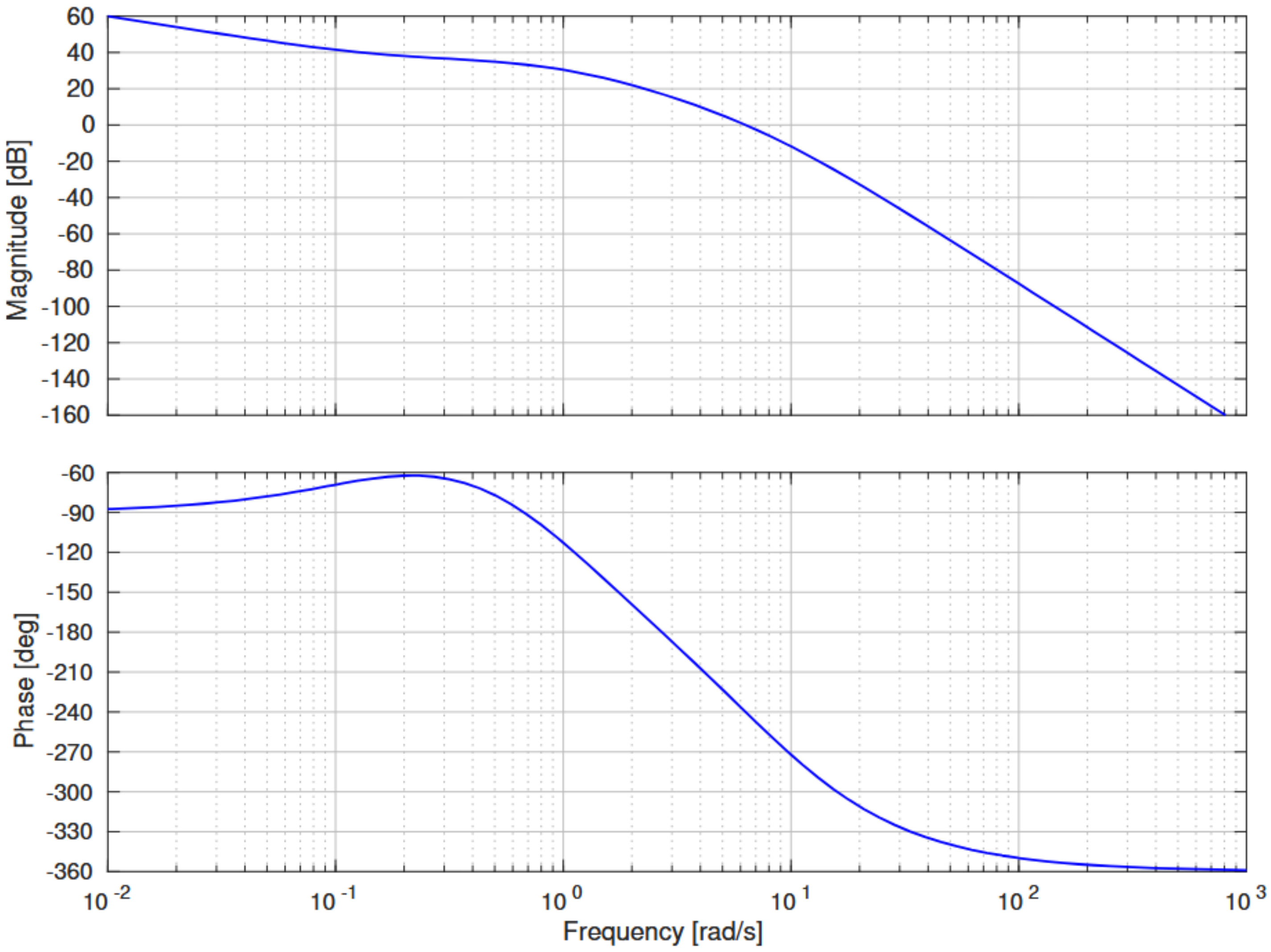
A) $GM \approx 18\text{ dB}$

B) $GM \approx 5\text{ dB}$

C) $GM \approx 90\text{ dB}$

D) $GM \approx -17\text{ dB}$

FS 2024



The GM is best described by:

A) $GM \approx 18\ \mathrm{dB}$

B) $GM \approx 5\ \mathrm{dB}$

C) $GM \approx 90\ \mathrm{dB}$

D) $GM \approx -17\ \mathrm{dB}$

Q&A Session / Done

Feedback

jschultev.github.io/personal_website/Feedback